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RESEARCH ARTICLE

Algebraic properties of periodic points to the additive cellular automata

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Abstract

This paper characterizes the periodic points of additive cellular automata such as, arbitrary union and arbitrary intersection, the relationship between periodic sets and the existence of additive cellular automata with cardinality of periodic points one and two and existence for higher cardinality more than two.

Keywords: Dynamical systems, Additive cellular automata, Period, Periodic point.

AMS Subject Classification: 37B15,37C25

Introduction

Knowing about the periodic properties of dynamical systems is a useful area of research and got attention after Sarkovski's paper (Sarkovski, 2019), mathematician became actively focusing on dynamical systems as the set of periods of periodic points of a linear operator (Akbar *et al*., 2019). The set of periods of periodic points of a toral automorphism (Kannan *et al*., 2011) and in (Pillai *et al*., 2010), Periodic points for onto cellular automata in (Boyle *and Kitchens*, 1999). Sets of periods of continuous maps on some metric spaces (Saradhi, 1997).

In this paper we are going to investigate the properties of $Per(F)$,$P_k(F)$ and $P(F)$

$$Per(F) = \{ m \in \mathbb{N} : m \text{ is the least positive number such that } F^m(x) = x \text{ in } A^{\mathbb{Z}} \text{ on } F: B^{\mathbb{Z}} \to B^{\mathbb{Z}}\} \quad (Eq.1)$$

$$P(F) = \{A \in A^{\mathbb{Z}} : \text{Every point in } A \text{ is a periodic point of } F: A^{\mathbb{Z}} \to A^{\mathbb{Z}} \} \quad (Eq.2)$$

$$P_k(F) = \{x \in A^{\mathbb{Z}}: F^k(x) = x\} \quad (Eq.3)$$

Which contains those points whose period divides k for the additive cellular automata in the specific form for $A = \{0,1,2,\ldots..l-1\}$ where $l \geq 2$.

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Preliminaries

Definition

(X, f) is called dynamical system where X is a topological space and f is a self-continuous map on X.

We consider (X, f) is a topological dynamical system for the rest of the article.

Definition

$\{x, f(x), f^3(x), \ldots..\}$ is called the trajectory of any point x belongs to X

Definition

The set of all distinct element of trajectory is called orbit

Denoted by $O(f, x) = \{f^m(x): m \in \mathbb{N} \cup \{0\}\}$.

Definition

x is said to periodic point if there is $m \in \mathbb{N}$ such that

$$f^m(x) = x \quad (Eq.4)$$

The least m is called the period of x (Holmgren, 2010) and in (Block and Coppel,2006) (Eq.5)

Here we get infinite number of natural numbers m, satisfying equation (Eq.4) and all are multiples of m.

Definition

A point of period one is called the fixed point fixed

Definition

The set of periods denoted by

$$PerF = \{ m \in \mathbb{N} : m \text{ is the least positive number such that } F^m(x) = x \text{ in } A^{\mathbb{Z}} \text{ on } F: B^{\mathbb{Z}} \to B^{\mathbb{Z}}\}$$

Periods have many applications as the points repeat with fixed time.

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Periodic properties for dynamical systems has been an interesting problem. This has been studied extensively studied in the literature.

Definition

For natural number $l \geq 1$, Let

$A = \{0,1,2,3,4 \ldots\ldots l-1\}(mod\ l)$

We call A as an alphabet and its elements as symbols. Let $\Sigma = A^{\mathbb{Z}}$ be the additive group of infinite two sided sequences of symbols in A and Let $\Sigma^+ = A^{\mathbb{N}}$ be the infinite one sided sequences which is also an additive group of infinite order.

We can define metric on these spaces which induces the same topology.

Given a one sided or two-sided sequence $c = (c_m)$, let $\sigma(c)$ be the sequence given by

$\sigma(c_m) = c_{m+1}$ (Eq.6)

This defines a continuous self map of both Σ and Σ^+, called the shift map.

The system (Σ, σ) is called the two-sided shift map and (Σ^+, σ) is called the full one-sided shift.

For the infinite two-sided sequence

$(x) \in A^{\mathbb{Z}}, x = \cdots x_{-2}x_{-1}.x_0x_1 \ldots\ldots$, where dot is the indication for x_0

Definition

A cellular automata is a dynamical system $(A^{\mathbb{Z}}, F)$, such that it commutes with shift map defined in (Eq.6)

$F \circ \sigma = \sigma \circ F$

Another definition is function F from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ a cellular automata if there exist

$r \in \mathbb{N}$ (radius) and a local rule

$f: A^{2r+1} \rightarrow A$ such that

$F(x)_i = f(x_{i-r}, \ldots.x_0 \ldots\ldots x_{i+r})$ (Eq.7)

For every $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

That in the infinite sequence function value of every $x_i\ in\ (x_i)_{i\in\mathbb{Z}}$ depends upon the neighborhood of x_{i-r} to x_{i+r}.

Both definitions are equivalent proved by G.A.Hedlund (Hedlund , 1969)

For Example $A = \{0,1,2,3,4,5\}$,

Define $F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ as $F(x)_i = 2x^2{}_{i-1}3x_i + 4x^3{}_{i+1}$.

Here $x_i\ in\ (x_i)_{i\in\mathbb{Z}}$ is depends upon neighborhood x_{i-1} and x_{i+1}.

Definition

Additive or linear cellular automata is one kind of cellular automata in which value of each $x_i\ in\ (x_i)_{i\in\mathbb{Z}}$ it the linear combination of its neighborhood values as the specified radius of cellular automata.

Let $l \geq 2$ be an integer and $A = \{0,1,2,3,4 \ldots\ldots l-1\}$

Additive cellular automata is a map from $F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$, for $b_i \in \mathbb{R}, -k \leq i \leq k$ which is of the form

$F(x)_i = b_{-k}x_{i-k} + b_{-k+1}x_{i-k+1} \ldots\ldots + b_0x_i \ldots\ldots b_{k-1}x_{i+k-1} + b_k(mod\ l)$ (Eq.8)

For example,

$A = \{0,1,2,3,4,5\}$,

Define $F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ as $F(x)_i = 2x_{i-1} - 3x_i + 4x_{i+1}$.

Here $x_i\ in\ (x_i)_{i\in\mathbb{Z}}$ is depends upon neighborhood x_{i-1} and x_{i+1}.

The difference between cellular automata and additive cellular automata is the linearity.

Results

Here we describe the basic properties periodic points of additive cellular automata defined in (Eq.8).

As we defined in (Eq.2) and (Eq.3) we can write

$P(F) = \cup_{k\in\mathbb{N}} P_k(F)$ (Eq.9)

So studying the properties of $P_k(F)$ is equivalent to the studying the properties of $P(F)$

We write the following lemma without proving as it is trivial.

***Lemma* 1:** I is an identity function from $A^{\mathbb{Z}}$ to itself and F is additive cellular automata as defined in (Eq.8) then the map $I - F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is a homomorphism.

Proof: We can prove it by considering for all $x, y \in A^{\mathbb{Z}}$

$(I - F)(x + y) = (I - F)x + (I - F)y$

***Lemma* 2:** $P_1(F)$ is a normal subgroup of $A^{\mathbb{Z}}$.

Proof: As every element $x \in P_1(F), F(x) = x$

$(I - F)(x) = \mathbf{0}$ (Here $\mathbf{0}$ is zero element in $A^{\mathbb{Z}}$)

That implies $P_1(F)$ is the kernal for the function $I - F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ and kernal is normal subgroup for $A^{\mathbb{Z}}$.

***Lemma* 3:** $P_k(F)$ is a normal subgroup of $A^{\mathbb{Z}}$.

Proof: Similarly, to the lemma 1 define function as $I - F^k: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ then $P_k(F)$ is the kernal for as $I - F^k$ and normal subgroup for $A^{\mathbb{Z}}$.

***Theorem* 1:** $P_1(F)$ contains at least one element for any additive cellular automata defined in (Eq.8).

Proof: This can be proved in two ways.

From Lemma 1, $P_1(F)$ is a normal sub group of $A^{\mathbb{Z}}$ and every normal subgroup of a additive group contains identity element that is zero element $\mathbf{0}$.

Here $\mathbf{0} = \cdots \ldots 00000.00000 \ldots\ldots\ldots \in A^{\mathbb{Z}}$

Another way is By the definition of Additive cellular automata which is linear combination of its neighbourhood values (Which are all zeros) is zero i.e $F(\mathbf{0}) = \mathbf{0}$

$\Rightarrow \mathbf{0} \in P_1(F)$.

***Theorem* 2:** Cardinality of $P_1(F)$ is denoted as $|P_1(F)|$ then $|P_1(F)| \geq 1$.

Proof: From Theorem 1 , $P_1(F)$ contains at least one element of $A^{\mathbb{Z}}$ that is zero element . So the cardinality is always greater than or equal to one.

***Theorem* 3:** $P_k(F)$ contains at least one element for any additive cellular automata defined in (Eq.8).

Proof: By the lemma 3, $P_k(F)$ is a normal sub group of $A^{\mathbb{Z}}$ and every normal subgroup of a additive group contains identity element that is zero element $\mathbf{0}$.

***Theorem* 4:** Cardinality of $P_k(F)$ is denoted as $|P_k(F)|$ then $|P_k(F)| \geq 1$.

Proof: Similar to the theorem 2 as , $P_k(F)$ is a normal sub group of $A^{\mathbb{Z}}$.

***Theorem* 5:** Every additive cellular automata has a periodic point .

Proof: As the definition of additive cellular automata defined in (Eq.8),

$F(\mathbf{0}) = \mathbf{0}$

So $\mathbf{0}$ is the fixed point which is equal to periodic point of period 1 .

***Theorem* 6:** Finite intersection and arbitrary intersection of $P_k(F)$ is non empty.

Proof: As every : $P_k(F)$ is having $\mathbf{0}$ element so finite and arbitrary intersection of $P_k(F)$ is non empty.

***Theorem* 7:** $P_1(F) \subseteq P_k(F)$ for every positive integer k.

Proof: For all $x \in P_1(F)$ then $F(x) = x$

$\Rightarrow F^k(x) = x$

So $x \in P_k(F)$

$\Rightarrow$: $P_1(F) \subseteq P_k(F)$

***Theorem* 8:** $|P_1(F)| \leq |P_k(F)|$ for any positive integer.

Proof: By the theorem 7, $P_1(F) \subseteq P_k(F)$

$\Rightarrow |P_1(F)| \leq |P_k(F)|$

***Theorem* 9:** $P_1(F)$ is the only subset such that $P_1(F) \subset P_k(F)$ for every prime integer k.

Proof: For prime k, $P_k(F)$ contains those elements whose period divides prime k. For prime k the divisors are 1 and itself.

So $P_1(F)$ is the only subset such that $P_1(F) \subset P_k(F)$ for every prime integer k.

***Theorem* 10:** $P_1(F)$ is the only subset such that $P_1(F) \subseteq P_k(F) \cap P_m(F), (k, m) = 1$ where GCD of k *and* m is 1.

Proof: By the definition of $P_k(F)$, it contains all the elements whose period divides k.

As the gcd of k *and* m is 1.

The elements common to $P_k(F)$ and $P_m(F)$ are $P_1(F)$ only.

So $P_1(F) \subseteq P_k(F) \cap P_m(F)$

***Theorem* 11:** $P_1(F)$ is the only subset such that $P_1(F) \subseteq P_k(F) \cap P_m(F), k, m$ are primes.

Proof: k, m are primes so their gcd is 1. We can apply the theorem 10.

***Theorem* 12:** The additive cellular automata is in the form of $F(x)_i = x_i (mod\ l)$ then $|Per(F)| = 1$

Proof: For $A = \{0,1,2,3,4 \dots \dots l-1\}$ and image of every element is same including zero element.

So every element is fixed that is period one. So set of periods contains only one number that is 1. That implies the cardinality of $Per(F)$ is 1.

In the case every point is periodic so $P(F) = A^{\mathbb{Z}}$.

***Theorem* 13:** The additive cellular automata is in the form of $F(x)_i = cx_i (mod\ l)$ then $|Per(F)| \leq 2$.

Proof: For any additive cellular automata zero element is the fixed. Now we will examine for the conditions period 2.

As specified in (Holmgren *et al.*, 2000), For l is a prime then $c(mod\ l)$ lies in the multiplicative group of $\{1,2,3,4 \dots \dots l-1\}(mod\ l)$ of order $l - 1$.

So Every element has some finite order s, Such that $c^s = 1(mod\ l)$.

$\Rightarrow F^s(x)_i = c^s x_i = 1\ (mod\ l)$.

If l is a prime then we get the cardinality of $Per(F)$ is 2.

In another case if $(c, l) = 1$.

Then c contains in the unitary group of $U(l)$, which contains all the positive integers less than l and coprime to l.

Here $U(l)$ has finite group. So, every element has some finite order s, Such that $c^s = 1(mod\ l)$.

$\Rightarrow F^s(x)_i = c^s x_i = 1\ (mod\ l)$.

In this case also we get cardinality of $Per(F)$ is 2.

In other cases, we can't guarantee the existence of the second period.

Discussion

The group nature of the periodic set of additive cellular automata was covered in this paper. We also discussed about different periodic sets' cardinality. Finding a periodic set with a given cardinality for the given additive cellular automata remains an open question. We can also talk about the periodic nature of general cellular automata, but this is a little more difficult because in some cases, zero may not be a fixed point, making it impossible for it to be the normal subgroup of the additive group of two-sided infinite sequences modulo.

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