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**RESEARCH ARTICLE**

# Coupled Fixed Point Theorems in Bipolar Parametric Metric Spaces with Applications to Coupled System of Differential Equations

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**ABSTRACT**

This paper investigates coupled fixed-point results in bipolar parametric metric spaces. The main objective is to establish the existence and uniqueness of a coupled fixed point for a covariant mapping under suitable contractive conditions. By constructing appropriate left and right iterative sequences, we show how the contraction condition leads to the Cauchy property of the resulting bisequences. Completeness of the space is then used to obtain their limits, and the contractive inequality helps establish that these limits form a coupled fixed point. A uniqueness argument is also presented to demonstrate that no other coupled fixed point can satisfy the same assumptions. A corollary is derived as a consequence of the principal theorem, and an illustrative example is provided to explain the applicability of the theoretical result. Finally, a coupled system of first-order differential equations is considered and transformed into an equivalent integral system. This formulation indicates how coupled fixed-point techniques can be used to investigate solutions of differential equations. The work highlights the role of completeness, covariant mappings, and contractive conditions in coupled fixed-point analysis.

***Mathematics Subject Classification**: 47H10; 54H25*

***Keywords:** Bipolar parametric metric space; coupled fixed point; covariant mapping; Coupled system of differential equations.*

## INTRODUCTION

The "Banach contraction principle" (1922) (Banach, 1922) is the starting point of metric fixed point theory, and its extensions and applications are collected in (Agarwal et al., 2001; (Granas & Dugundji, 2013; Kirk & Shahzad, 2014); a comparison of the many contractive conditions is given in (Rhoades, 1977). The metric axioms have since been relaxed in several directions, for instance b-metric spaces (Bakhtin, 1989) (Bakhtin, 1989; Czerwik, 1993), multiplicative metric spaces (Bashirov et al., 2008; Mutlu et al., 2020), and parametric metric spaces, in which the distance depends on a parameter (Hussain et al., 2015).

A different generalization measures the distance between points of two different sets. Mutlu (2016) introduced bipolar metric spaces for this purpose and proved Banach-type theorems for covariant and contravariant mappings; further contractive conditions in this setting were studied in (Rawat et al., 2022; Gürdal et al., 2020; Mutlu et al., 2020). Extended bipolar b-metric spaces (Ganeswari & Leema Maria Prakasam, 2026), bipolar b-multiplicative metric spaces (Mercy Arokia Rani & Leema Maria Prakasam, 2024) and modular bipolar metric spaces (Maria Merlin & Leema

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Maria Prakasam, 2025) enlarge the framework further, and Hardy–Roger's type contractions are treated in (Mercy Arokia Rani & Leema Maria Prakasam, 2025).

Coupled fixed points were introduced by Guo and Lakshmikantham (1987) and were developed in partially ordered metric spaces in (Bhaskar & Lakshmikantham, 2006; Lakshmikantham & Ćirić, 2009. In bipolar metric spaces, coupled fixed point theorems were obtained in Mutlu et al., (2017) and Kishore et al., (2021), and common coupled fixed-point theorems in a parametric-type setting appear in Maria Merlin & Leema Maria Prakasam, (2026). Pasha (2024) combined the bipolar and the parametric ideas by introducing the bipolar parametric metric space (BPPMS), and related bipolar spaces were studied in (Kumar et al., 2024; Mani et al., 2023).

In this paper we prove a coupled fixed-point theorem in complete BPPMS. If $F: (P^2, Q^2) \rightrightarrows (P, Q)$ is a covariant mapping satisfying $d(F(a,b), F(p,q), C) \leq k\, d(a,p,C) + l\, d(b,q,C)$ with $k + l < 1$, then $F$ has a unique coupled fixed point (Theorem 3.3).

The paper is organized as follows. Section 2 recalls the definitions, Section 3 contains the main result and an example, Section 4 the application, Section 5 a discussion, and Section 6 the conclusion.

## PRELIMINARIES

***Definition. 2.1***: (Mercy Arokia Rani & Leema Maria Prakasam, 2025; Pasha et al., 2024)**:** Suppose $d: P \times Q \times (0, \infty) \to \mathbb{R}^+$ is a function defined on two non empty sets $P$ and $Q$ such that.

(1) If $d(p,q,C) = 0$ for all $C > 0$ then $p = q$, for all $(p,q) \in P \times Q$.

(2) If $p = q$, then $d(p,q,C) = 0$, for all $C > 0$ and $(p,q) \in P \times Q$

(3) $d(p,q,C) = d(q,p,C)$, for all $C > 0$ and $p, q \in P \cap Q$

(4) $d(p_1, q_2, C) \leq d(p_1, q_1, C) + d(p_2, q_1, C) + d(p_2, q_2, C)$, for all $C > 0$, $p_1, p_2 \in P$ and $q_1, q_2 \in Q$.

The triplet $(P, Q, d)$ is called a BPPMS.

***Example. 2.2:*** Let $P = [0,1]$ and $Q = [1,2]$, and define

$d: P \times Q \times (0, \infty) \to \mathbb{R}^+$ by $d(p,q,C) = \frac{|p-q|}{1+C}$.

Then $(P, Q, d)$ is a complete BPPMS.

***Definition. 2.3*** (Maria Merlin & Leema Maria Prakasam, 2025; Mutlu, 2016)**:** Let $(P_1, Q_1, d_1)$ and $(P_2, Q_2, d_2)$ be BPPMSs and $F: P_1 \cup Q_1 \to P_2 \cup Q_2$ be a function. If $F(P_1) \subseteq P_2$ and $F(Q_1) \subseteq Q_2$, then $F$ is termed as a covariant map and this is written as $F: (P_1, Q_1, d_1) \rightrightarrows (P_2, Q_2, d_2)$. If $F$ is mapping from $(P_1, Q_1, d_1)$ to $(Q_2, P_2, \overline{d}_2)$, then $F$ is termed as a contravariant and this is denoted as $F: (P_1, Q_1, d_1) \rightleftarrows (P_2, Q_2, d_2)$.

**DEFINITION. 2.4** (Ganeswari & Leema Maria Prakasam, 2026; Mutlu, 2016)**:** Let $(P, Q, d)$ be a BPPMS. Then

(z1) The points of the sets $P$, $Q$, and $P \cap Q$ are referred to as left, right, and central points, respectively. A sequence on $(P, Q, d)$ that consists solely of left, right, or central points is termed a left, right, or central sequence.

(z2) A left sequence $\{a_n\}$ converges to a right point $q$ if and only if for every $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that $d(a_n, q, C) < \varepsilon$ for all $n \geq n_0$ and $C > 0$.

Similarly, a right sequence $\{p_n\}$ converges to a left point $a$ if and only if there exists a $n_0 \in \mathbb{N}$ such that $d(a, p_n, C) < \varepsilon$ for all $n \geq n_0$ and $C > 0$.

**DEFINITION. 2.5** (Mercy Arokia Rani & Leema Maria Prakasam, 2024; Mutlu, 2016): Let $(P, Q, d)$ be a BPPMS.

(i) A sequence $(\{a_n\}, \{p_n\}) \subseteq P \times Q$ is called a bisequence on $(P, Q, d)$.

(ii) The bisequence $(\{a_n\}, \{p_n\})$ is said to be convergent if both $\{a_n\}$ and $\{p_n\}$ are convergent. This bisequence is said to be biconvergent if $\{a_n\}$ and $\{p_n\}$ converge to the same point $u \in P \cap Q$.

(iii) $(\{a_n\}, \{p_n\})$ is a bisequence for $(P, Q, d)$ is known as a Cauchy bisequence (CBS) if, for every $\varepsilon > 0$, we can locate a number $n_0 \in \mathbb{N}$, which, for all positive integers $n, m \geq n_0$, $C > 0$, $d(a_n, p_m, C) < \varepsilon$, is called a CBS.

***Definition. 2.6*** (Khan et al., 1984): Let $\mathbb{A} = \{\mathcal{A} | \mathcal{A}: [0, \infty) \to [0, \infty)\}$ be a family of altering distance functions such that $\mathcal{A}$ is continuous and monotone non-decreasing functions; $\mathcal{A}(\varepsilon) = 0 \Leftrightarrow \varepsilon = 0$;

## NEW RESULTS IN COUPLED FIXED-POINT THEOREMS ON BIPOLAR PARAMETRIC METRIC SPACES

**DEFINITION. 3.1:** Let $(P, Q, d)$ be a bipolar parametric metric space. $F: (P^2, Q^2) \rightrightarrows (P, Q)$ be a covariant mapping $(p,q) \epsilon P^2 \cup Q^2$ is said to be coupled fixed point of $F$ if $F\ (a,b) = a$ *and* $F(b,a) = b$.

***Definition. 3.2:*** The mapping $d: P \times Q \times [0, \infty) \to [0, \infty)$ is assumed to possess the following properties. Consider two nonempty sets, $P$ and $Q$ and

C >0

$(B1)$ For every $(p,q) \in P \times Q$ , $p = q$ *if* $d\ (p,q,C) = 0$.

$(B2)$ For every $(p,q) \in P \times Q, d(p,q,C) = 0 \; if \; p = q.$

$(B3)$ If $d(p,q,C) = d(p,q,C)$, then for every $p,q \in P \cap Q$

$(B4)$ For every $p_1, p_2 \in P \; and \; q_1, q_2 \in Q, d(p_1,q_2,C) \leq d(p_1,q_1,C) +$

$$d(p_1,q_1,C) + d(p_2,q_2,C)$$

After defining the triple $(P,Q,d)$ as a bipolar-metric space, the bipolar metric of the pair $(P,Q)$ is defined as the mapping d.

Note: A bipolar parametric metric space is a generalization of metric space $(P,Q,d)$ where distance is measured between elements of two distinct sets $P$ and $Q$ using the function $d: P \times Q \times [0,\infty) \to [0,\infty)$ is satisfying special axioms regarding distance.

***Theorem. 3.3:***

Let $(P,Q,d)$ be a complete bipolar parametric metric space. $F: (P^2, Q^2) \rightrightarrows (P,Q)$ be a covaraiant mapping and $k,l$ be non-negative constants. If $F$ satisfies the condition

$$d(F(a,b,C),F(p,q,C) \leq k\, d(a,p,C) + ld(b,q,C), \quad k+l<1 \qquad (1)$$

$(a,b) \epsilon P, p.q \epsilon Q, then \; F: P^2 \cup Q^2 \to P \cup Q$ has a unique coupled fixed point.

***Proof:***

Let $a_0, b_0 \in P$ and $p_0, q_0 \in Q. We\ take \;\; a_1, b_1 \in P$ and $p_1, q_1 \in Q.$ with $F(a_0,b_0) = a_1 and\ F(b_0,a_0) = b_1, F(p_0,q_0) = p_1, F(q_0,p_0) = q_1$ and similarly we take $a_2, b_2 \in P$ and $p_2, q_2 \in Q.$ with $F(a_1,b_1) = a_2$ and $F(b_1,a_1) = b_2, F(p_1,q_1) = p_2, F(q_1,p_1) = q_2.$

In this way, we obtain a bisequences $a_n, b_n \in P$ and $p_n, q_n \in Q.$ with $F(a_n,b_n) = a_{n+1} and\ F(b_n,a_n) = b_{n+1},$

$$F(p_n,q_n) = p_{n+1}, F(q_n,p_n) = q_{n+1.}$$

For all $n \in \mathbb{N}^+$. Let $k+l = \lambda$. From (1), we get

$$d(a_n,p_{n+1,}C) = d(F(a_{n-1},b_{n-1},C),F(p_n,q_n,C) \qquad (2)$$

$$\leq k\, d(a_{n-1},p_n,C) + ld(b_{n-1},q_n,C) \quad and$$

$$d(b_n,q_{n+1,}C) = d(F(b_{n-1},a_{n-1},C),F(q_n,p_n,C) \qquad (3)$$

$$\leq k\, d(b_{n-1},q_n,C) + ld(a_{n-1},p_n,C)$$

For all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$e_n = d(a_n,p_{n+1},C) + d(b_n,q_{n+1},C)$$

For all $n \in \mathbb{N}^+$ .Combining (2) and (3), We observe that

$$e_n = d(a_n,p_{n+1},C) + d(b_n,q_{n+1},C)$$

$$\leq k\, d(a_{n-1},p_n,C) + ld(b_{n-1},q_n,C) + k\, d(b_{n-1},q_n,C) + ld(a_{n-1},p_n,C)$$

$$= (k+l)\, d(a_{n-1},p_n,C) + d(b_{n-1},q_n,C) = \lambda e_{n-1}.$$

Then we get

$$0 \leq e_n \leq \lambda e_{n-1} \leq \lambda^2 e_{n-1} \leq \dots\dots\dots\dots \leq \lambda^n e_{0.} \qquad \rightarrow(4)$$

On the other hand

$$d(a_{n+1},p_{n,}C) = d(F(a_n,b_n,C),F(p_{n-1},q_{n-1},C)$$

$$\leq k\, d(a_n,p_{n-1},C) + ld(b_n,q_{n-1},C) \qquad \rightarrow(5)$$

$$d(b_{n+1},q_{n,}C) = d(F(b_n,a_n,C),F(q_{n-1},p_{n-1},C)$$

$$\leq k\, d(b_n,q_{n-1},C) + ld(a_n,p_{n-1},C) \qquad \rightarrow(6)$$

For all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$s_n = d(a_{n+1},p_n,C) + d(b_{n+1},q_n,C)$$

For all $n \in N^+$ .Combining (5) and (6), We observe that

$$s_n = d(a_{n+1},p_n,C) + d(b_{n+1},q_n,C)$$

$$\leq k\, d(a_n,p_{n-1},C) + ld(b_n,q_{n-1},C) + k\, d(b_n,q_{n-1},C) + ld(a_n,p_{n-1},C)$$

$$= (k+l)\, d(a_n,p_{n-1},C) + d(b_n,q_{n-1},C) = \lambda s_{n-1}.$$

Then we get

$$0 \leq s_n \leq \lambda s_{n-1} \leq \lambda^2 s_{n-1} \leq \dots\dots\dots\dots\dots\dots\dots \leq \lambda^n s_{0.} \qquad \rightarrow(7)$$

Moreover,

$$d(a_n,p_{n,}C) = d(F(a_{n-1},b_{n-1},C),F(p_{n-1},q_{n-1},C)$$

$$\leq k\, d(a_{n-1},p_{n-1},C) + ld(b_{n-1},q_{n-1},C) \qquad \rightarrow(8)$$

$$d(b_n,q_{n,}C) = d(F(b_{n-1},a_{n-1},C),F(q_{n-1},p_{n-1},C)$$

$$\leq k\, d(b_{n-1},q_{n-1},C) + ld(a_{n-1},p_{n-1},C) \qquad \rightarrow(9)$$

For all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$t = d(a_n,p_n,C) + d(b_n,q_n,C)$$

For all $n \in \mathbb{N}^+$ .Combining (8) and (9), We observe that

$$t_n = d(a_n,p_n,C) + d(b_n,q_n,C)$$

$$\leq k\, d(a_{n-1},p_{n-1},C) + ld(b_{n-1},q_{n-1},C) + k\, d(b_{n-1},q_{n-1},C) + ld(a_{n-1},p_{n-1},C)$$

$$= (k+l)\, d(a_{n-1},p_{n-1},C) + d(b_{n-1},q_{n-1},C) = \lambda t_{n-1}.$$

Then we get

$$0 \le t_n \le \lambda t_{n-1} \le \lambda^2 t_{n-1} \le \dots\dots\dots\dots \le \lambda^n t_0. \quad \rightarrow (10)$$

Using the property (B3), we get $d(a_n, p_m, C) \le d(a_n, p_{n+1}, C) + d(a_{n+1}, p_{n+1}, C) + \dots\dots\dots\dots + d(a_{m-1}, p_m, C)$

$$d(b_n, q_m, C) \le d(b_n, q_{n+1}, C) + d(b_{n+1}, q_{n+1}, C) + \dots\dots\dots\dots + d(b_{m-1}, q_m, C) \quad (11)$$

$$d(a_m, p_n, C) \le d(a_m, p_{m-1}, C) + d(a_{m-1}, p_{m-1}, C) + \dots\dots\dots\dots + d(a_{n+1}, p_n, C)$$

$$d(b_m, q_n, C) \le d(b_m, q_{m-1}, C) + d(b_{m-1}, q_{m-1}, C) + \dots\dots\dots\dots + d(b_{n+1}, q_n, C) \quad (12)$$

For each $n, m \in \mathbb{N}^+ \text{ and } n < m$. Then from (4), (7), (10), (11) and (12), we have

$d(a_n, p_m, C) + d(b_n, q_m, C)$

$\le (d(a_n, p_{n+1}, C) + d(b_n, q_{n+1}, C)) + (d(a_{n+1}, p_{n+1}, C) + d(b_{n+1}, q_{n+1}, C)) + \cdots\dots\dots\dots + d(a_{m-1}, p_m, C) + + d(b_{m-1}, q_m, C))$

$\le e_n + t_{n+1} + e_{n+1+\dots} + t_{m-1} + e_{m-1}$

$\le \lambda^n e_0 + \lambda^{n+1} t_0 + \lambda^{n+1} e_0 + \dots\dots + \lambda^{m-1} t_0 + \lambda^{m-1} e_0$

$$\le \frac{\lambda^n}{1-\lambda} e_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 \quad \rightarrow (13)$$

$d(a_m, p_n, C) + d(b_m, q_n, C)$

$\le (d(a_m, p_{m-1}, C) + d(b_m, q_{m-1}, C)) + (d(a_{m-1}, p_{m-1}, C) + d(b_{m-1}, q_{m-1}, C)) + \dots\dots\dots\dots + d(a_{n+}, p_n, C) + + d(b_{n+1}, q_n, C))$

$\le s_{m-1} + t_{m-1} + \cdots + s_{n+1} + t_{n+1} + s_n$

$\le \lambda^{m-1} s_0 + \lambda^{m-1} t_0 + \dots + \lambda^{n+1} s_0 + \lambda^{n+1} t_0 + \lambda^n s_0$

$$\le \frac{\lambda^n}{1-\lambda} s_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 \quad \rightarrow (14)$$

For $n < m$. Since, for arbitrary $\epsilon > 0$, there exists $n_0$ such that

$\frac{\lambda^n}{1-\lambda} e_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 < \frac{\epsilon}{3}$ and $\frac{\lambda^n}{1-\lambda} s_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 < \frac{\epsilon}{3}$. From (13) and (14), we have

$$d(a_n, p_m, C) + d(b_n, q_m, C) < \frac{\epsilon}{3} \quad \rightarrow (15)$$

for each $n, m \ge n_0$. Then $(a_n, p_m, C)$ and $(b_n, q_m, C)$ are Cauchy bisequences. Because of completeness of $(P, Q, d)$, there exists $a, b \in P$ and $p. q \in Q$ with

$\lim_{n\to\infty} a_n = p$, $\lim_{n\to\infty} b_n = q$, $\lim_{n\to\infty} p_n = a$ and $\lim_{n\to\infty} q_n = b$

Then there exists $n_1 \in \mathbb{N}$ with

$d(a_n, p, C) < \frac{\epsilon}{3}$, $d(b_n, q, C) < \frac{\epsilon}{3}$, $d(b, q_n, C) < \frac{\epsilon}{3}$ and $d(p, a_n, C) < \frac{\epsilon}{3}$. So from (1), we have

$$d(F(a,b), p, C) \le d(F(a,b), p_{n+1}, C) + d(a_{n+1}, p_{n+1}, C) + d(a_{n+1}, p, C)$$

$= d(F(a,b), F(p_n, q_n, C) + d(a_{n+1}, p_{n+1}, C) + d(a_{n+1}, p, C)$

$\le k\, d(a, p_n, C) + l\, d(b, q_n, C) + d(a_{n+1}, p_{n+1}, C) + d(a_{n+1}, p, C)$

$\le k\frac{\epsilon}{3} + l\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$= \lambda\frac{\epsilon}{3} + 2\frac{\epsilon}{3} < \varepsilon$

For all $n \in \mathbb{N}$ and $\lambda < 1$. Then $d(F(a,b), p, C) = 0$.

$Hence, F(a, b, C) = p$. Similarly, we get

$F(b, a) = q, F(p, q) = a$ and $F(q, p) = b$. On the other hand, from (15),

we get

$$d(a, p, C) = \lim_{n\to\infty} d(a_n, p_n, C) = 0$$

$d(b, q, C) = \lim_{n\to\infty} d(b_n, q_n, C) = 0$. So $a = p$ and $b = q$.

Therefore, $(a, b, C) \in P^2 \cup Q^2$ is a coupled fixed point of $F$.

Now, to show the uniqueness, we begin by taking another coupled fixed point $(a*, b*, C) \in P^2 \cup Q^2$. $If\ (a*, b*, C) \in P^2$, then we get

$$d(a^*, a, C) = d\big(F(a^*, b^*, C), F(a, b, C)\big) \le k\, d(a^*, a, C) + l\, d(b^*, b, C)$$

and $d(b^*, b, C) = d\big(F(b^*, a^*, C), F(b, a, C)\big) \le k\, d(b^*, b, C) + l\, d(a^*, a, C)$

$$d(a^*, a, C) + d(b^*, b, C) \le \lambda\, d(a^*, a, C) + d(b^*, b, C) \quad \rightarrow (16)$$

Since $\lambda < 1$. This means that $d(a^*, a, C) + d(b^*, b, C) = 0$, we obtain that $a^* = a$ and $b^* = b$. $Similarly$, if $d(a^*, b^*, C) \in Q^2$ we obtain that $a^* = a$ and $b^* = b$.

Then $(a, b)$ is a unique coupled fixed point of F.

***Corollary 3.4:***

Let $(P, Q, d)$ be a complete bipolar parametric metric space. F: $(P^2, Q^2) \rightrightarrows (P, Q)$ be a covaraiant mapping and $k, l$ be non-negative constants. If $F$ satisfies the condition

$d(F(a,b,C), F(p,q,C) \le \frac{k}{2}(d(a,p,C) + ld(b,q,C)), \quad k < 1$

$(a, b) \epsilon P, p. q \epsilon Q, then\ F: P^2 \cup Q^2 \to P \cup Q$ has a unique coupled fixed point.

***Example. 3.5:*** Let $(P, Q, d)$ be the complete BPPMS of Example 2.2, that is $P = [0,1]$, $Q = [1,2]$ and $d(p,q,C) = \frac{|p-q|}{1+C}$.

Define $F: P^2 \cup Q^2 \to P \cup Q$ by $F(x,y) = \frac{2x+y+5}{8}$.

If $x, y \in P = [0,1]$, then $\frac{5}{8} \le F(x,y) \le 1$,

so $F(P^2) \subseteq P$. If $x, y \in Q = [1,2]$, then $1 \le F(x,y) \le \frac{11}{8}$, so $F(Q^2) \subseteq Q$.

Let $(a,b) \in P^2$ and $(p,q) \in Q^2$. Then

$$F(a,b) - F(p,q) = \frac{2(a-p) + (b-q)}{8},$$

so that

$$|F(a,b) - F(p,q)| \le \frac{1}{4}|a-p| + \frac{1}{8}|b-q|.$$

Dividing by $1 + C$ we obtain

$$d(F(a,b), F(p,q), C) \le \frac{1}{4}\, d(a,p,C) + \frac{1}{8}\, d(b,q,C),$$

with $k = \frac{1}{4}$, $l = \frac{1}{8}$ and $k + l = \frac{3}{8} < 1$.

All hypotheses of Theorem 3.3 hold, so $F$ has a unique coupled fixed point. To find it, solve $F(x,y) = x$ and $F(y,x) = y$:

$$y = 6x - 5, \qquad x = 6y - 5.$$

Substituting the first into the second gives $x = 36x - 35$, hence $x = 1$ and $y = 1$.

Thus $(1,1) \in P \cap Q$ is the unique coupled fixed point, and indeed

$$F(1,1) = \frac{8}{8} = 1.$$

Take $a_0 = b_0 = 0 \in P$ and $p_0 = q_0 = 2 \in Q$. Then $a_n = b_n, p_n = q_n$

and $a_{n+1} = \frac{3a_n+5}{8}$, $p_{n+1} = \frac{3p_n+5}{8}$, which gives

$$a_n = b_n = 1 - (\frac{3}{8})^n, \qquad p_n = q_n = 1 + (\frac{3}{8})^n.$$

Hence $d(a_n, p_n, C) = \frac{2(\frac{3}{8})^n}{1+C} \to 0$, and both sequences converge to the central point 1, the coupled fixed point of $F$. Moreover, with $e_n = d(a_n, p_{n+1}, C) + d(b_n, q_{n+1}, C)$ we get $e_n = \lambda^n e_0$ for $\lambda = k + l = \frac{3}{8}$, which agrees with the estimate used in the proof of Theorem 3.3.

## APPLICATION TO A COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS

Consider the following system of first-order differential equations with initial conditions:

$$\begin{aligned} x'(t) &= (1/10)[x(t) + y(t)], \quad x(0) = 1, \\ y'(t) &= (1/10)[y(t) + x(t)], \quad y(0) = 1. \end{aligned} \tag{4.1}$$

We represent the system as a coupled fixed-point problem. The theorem-based conclusion is valid provided the function-space, distance, and mapping hypotheses in Theorem 3.3 are all satisfied.

Integrating the first differential equation from 0 to t and using $x(0) = 1$ gives

$$x(t) = 1 + \left(\frac{1}{10}\right)\int_0^t [x(s) + y(s)]ds. \tag{4.2}$$

Similarly, using y(0) = 1, we obtain

$$y(t) = 1 + \left(\frac{1}{10}\right)\int_0^t [y(s) + x(s)]ds. \tag{4.3}$$

Thus, the differential system is equivalent to the integral system (4.2)–(4.3).

Let $X = C([0,1], \mathbb{R})$, equipped with the supremum norm

$$||u||\infty = \sup_{\{t\in[0,1]\}} |u(t)|.$$

Define T : X × X → X by

$$T(u,v)(t) = 1 + \left(\frac{1}{10}\right)\int_0^t [u(s) + v(s)]ds. \tag{4.4}$$

Since $u$ and $v$ are continuous, the integrand is continuous; hence $T(u,v)$ is continuous and $T$ is well-defined.

Take u, v, p, q ∈ X. Then

$$|T(u,v)(t) - T(p,q)(t)| = \left(\frac{1}{10}\right)\left|\int_0^t [(u(s) - p(s)) + (v(s) - q(s))]ds\right|.$$

By the triangle inequality,

$$|T(u,v)(t) - T(p,q)(t)| \le (1/10)\int_0^t (|u(s) - p(s)| + |v(s) - q(s)|)\, ds$$

Since $t \in [0,1]$,

$$|T(u,v)(t) - T(p,q)(t)| \le \left(\frac{1}{10}\right)(||u - p||\infty + ||v - q||\infty).$$

Taking the supremum over $t$ gives

$$||T(u,v) - T(p,q)||\infty \le \left(\frac{1}{10}\right)||u - p||\infty + \left(\frac{1}{10}\right)||v - q||\infty. \tag{4.5}$$

This is a coupled contraction estimate with $k = l = \frac{1}{10}$, so $k + l = \frac{1}{5} < 1$. For a formal application of Theorem 3.3, this estimate must be expressed in the BPPMS distance and the required covariant mapping structure must be verified.

Subtracting the second integral equation from the first gives

$x(t) - y(t) = 0$. Hence $x(t) = y(t)$ for all $t \in [0,1]$.

Substituting into the first differential equation yields

$$x'(t) = \left(\frac{1}{5}\right)x(t), \quad x(0) = 1. \quad (4.6)$$

Separating variables gives $\frac{dx}{x} = \frac{dt}{5}$. Integrating, $\ln|x(t)| = \frac{t}{5} + C$.

From $x(0) = 1$, we obtain $C = 0$. Therefore

$$x(t) = e^{\frac{t}{5}}. \quad (4.7)$$

Since $y(t) = x(t)$, we also have

$$y(t) = e^{\frac{t}{5}}. \quad (4.8)$$

For $x(t) = y(t) = e^{\frac{t}{5}}$, we have $x'(t) = y'(t) = \left(\frac{1}{5}\right)e^{\frac{t}{5}}$. Also,

$$\left(\frac{1}{10}\right)[x(t) + y(t)] = \left(\frac{1}{10}\right)\left[2e^{\frac{t}{5}}\right] = \left(\frac{1}{5}\right)e^{\frac{t}{5}}.$$

Thus, both differential equations are satisfied. The initial conditions

$x(0) = y(0) = 1$ also hold.

The system has the solution $x(t) = y(t) = e^{\frac{t}{5}}$ on $[0,1]$.

## DISCUSSION

The results obtained in this study illustrate the role of contractive mappings in establishing coupled fixed-point theorems in bipolar parametric metric spaces. The proposed framework considers the interaction between two sets through a covariant mapping and provides a systematic approach to the existence and uniqueness of coupled fixed points. In this setting, the contractive condition serves as the principal tool for controlling the behaviour of the iterative sequences.

Theorem 3.3 establishes the main theoretical result of the paper. Its proof demonstrates how suitable distance estimates can be used to study the convergence of the constructed bisequences. The use of completeness connects this convergence process with the existence of a coupled fixed point. The uniqueness argument further strengthens the result by showing that the coupled fixed point is determined uniquely under the prescribed hypotheses. Thus, the theorem brings together the contractive property, convergence, and completeness in a single mathematical argument.

Corollary 3.4 provides a consequence of the principal theorem and shows how the same fixed-point framework can be applied under the corresponding assumptions. The illustrative example in Example 3.5 gives a more direct understanding of the abstract result. By defining a specific mapping and solving the associated coupled equations, the unique coupled fixed point is identified explicitly. The example also provides an opportunity to examine the iterative process and observe its convergence towards the fixed point.

The differential-equation application presents another mathematical setting in which coupled fixed-point methods are relevant. The given system is transformed into an equivalent integral formulation by using the initial conditions. The corresponding integral operator is then studied through estimates involving the supremum norm. This approach demonstrates the connection between operator-based fixed-point techniques and the analysis of coupled differential systems.

Taken together, the theoretical results and the illustrative constructions show how the proposed framework can be used to investigate coupled fixed-point problems. The study also provides a starting point for examining more general contractive conditions and other nonlinear problems within bipolar parametric metric spaces.

## CONCLUSION

This work develops coupled fixed-point results in bipolar parametric metric spaces and establishes an existence and uniqueness theorem for a covariant mapping under suitable contractive conditions. The principal contribution lies in the use of iterative sequences and completeness to obtain the coupled fixed point and establish its uniqueness.

The results are supported by a corollary and an explicit example, which illustrate the theoretical framework and its mathematical implementation. In particular, the example identifies the coupled fixed point by solving the associated equations and demonstrates the convergence of the selected iterative sequences.

The study also considers a coupled system of first-order differential equations and expresses it as an equivalent system of integral equations. This formulation illustrates the relevance of coupled fixed-point techniques in the study of nonlinear differential systems. The work therefore provides a basis for further investigations into fixed-point theory in generalized metric structures.

Future research may focus on extending the present results to other classes of bipolar metric spaces, developing alternative contractive conditions, and investigating applications to nonlinear integral equations and boundary value problems. These directions may further broaden

the scope of coupled fixed-point theory and provide additional opportunities for its application in nonlinear analysis.

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