The Scientific Temper
SEPTEMBER 2026 Vol. 17 (9): 6981-6990
ISSN: 0976-8653, E-ISSN: 2231-6396
DOI: 10.58414/SCIENTIFICTEMPER.2026.17.9.2539
A WEB OF SCIENCE JOURNAL

RESEARCH ARTICLE

Some Common Fixed-Point Theorems in Extended Bipolar b-Metric Spaces and Applications

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ABSTRACT

In this paper, we prove some common fixed-point theorems in extended bipolar b-metric spaces in contravariant mappings and their applications under generalized rational type contractions. Two corollaries are derived as special cases of the main theorems, and an illustrative application to homotopy theory is given to demonstrate that a fixed point persists under continuous deformation between two contravariant mappings.

Mathematics Subject Classification: 47H10; 54H25

Keywords: *Bipolar metric space, b-Metric space, Extended b-Metric space, Common fixed point, Rational contraction, Contravariant mapping, Homotopy.*

INTRODUCTION

Fixed point theory sits at the heart of nonlinear analysis, mainly because contraction mappings give a clean way to guarantee that a solution exists and is unique. "Banach's (1922) theorem" remains the starting point for almost every generalization that came after it, guaranteeing a unique fixed point in any complete metric space under a simple contractive condition.

Not long after, "Bakhtin (1989) and Czerwik (1993)" loosened the triangle inequality itself, giving rise to b-metric spaces a small change that opened up a much wider class of spaces where distance relationships could be studied. "Kamran et al. (2017)" pushed this further by letting the constant in the triangle inequality depend on the points involved, which is how extended b-metric spaces came about. This point dependent control turned out to be genuinely useful it let researchers model systems that neither ordinary metric spaces nor plain b-metric spaces could handle comfortably.

A good deal of work has built on this idea since. "Alqahtani et al. (2019)" worked out rational-type inequalities in the extended b-metric setting, "Shatanawi and Shatnawi (2023)" introduced new contraction types in the same space, and "Javed et al. (2022)" carried the theory over to orthogonal extended b-metric spaces. On a separate track, the study of integral equations and inclusions has leaned heavily on fixed point arguments papers (8 - 23) give a sense of how broadly this approach has been applied and generalized contractions have also proved useful for analyzing stability in dynamical systems (24 – 28).

Bipolar metric spaces take a different route to generalization instead of relaxing the triangle inequality, they change what a “point” means, allowing distances to be measured between two disjoint sets rather than within a single set. This has proved especially natural for problems where

Received: 5th September, 2026 **Accepted: 7th September, 2026** **Published: 28th September, 2026**

two different types of objects interact think of covariant and contravariant mappings moving points between the two sides. Bipolar b-metric spaces, combining this idea with the relaxed triangle inequality of b-metric spaces, were introduced only recently. Since then the area has moved quickly within the last year alone, researchers have proposed extended bipolar parametric b-metric spaces and extended bipolar fuzzy b-metric spaces, applying them to problems as varied as fractional blood flow models and Meir–Keeler type contractions.

In this paper we take a more direct route into the same territory. Rather than layering a parametric or fuzzy structure on top of the bipolar b-metric, we extend it the same way Kamran et al. extended ordinary b-metric spaces through a control function ψ that depends on the points themselves and call the resulting structure an Extended Bipolar b-Metric space. Within this space we prove two common fixed-point theorems for contravariant mappings satisfying generalized rational-type contractive conditions, extending the Banach, Kannan, Jaggi and Khan contractions to this setting, together with two corollaries that follow as special cases. We close by applying these results to homotopy theory, showing that a fixed point persists throughout a continuous deformation between two mappings a concrete illustration of how the abstract theory connects back to problems in topology and analysis.

PRELIMINARIES

Definition 1.1: (16)

Let $U, V \neq \phi$ and $d: U \times V \to [0, +\infty)$ be a mapping satisfying the following properties:

i) $\mu = \gamma \ if \ d(\mu, \gamma) = 0$;
ii) $d(\mu, \gamma) = 0 \ if \ \mu = \gamma$;
iii) $d(\mu, \gamma) = d(\gamma, \mu), if \ \mu, \gamma \in U \cap V$
iv) $d(\mu_1, \gamma_2) \leq d(\mu_1, \gamma_1) + d(\mu_2, \gamma_1) + d(\mu_2, \gamma_2) \ for \ all \ \mu_1, \mu_2 \in U \ and \ \gamma_1, \gamma_2 \in V$.

Then the mapping d is called a bipolar metric on the pair (U, V) and the triple (U, V, d) is called a bipolar metric space.

Definition 1.2:

Let (U_1, V_1) and (U_2, V_2) be two pair of sets. A map $W: U_1 \cup V_1 \to U_2 \cup V_2$ is called

i) Covariant if $W(U_1) \subseteq U_2$ and $W(V_1) \subseteq V_2$ and it is denoted as $W: (U_1, V_1) \rightrightarrows (U_2, V_2)$.
ii) Contravariant if $W(U_1) \subseteq V_2$ and $W(V_1) \subseteq U_2$ and it is denoted as $W: (U_1, V_1) \rightleftarrows (U_2, V_2)$.

Definition 1.3:

Let (U, V, d) be a bipolar metric space. Then

a) $U = set \ of \ left \ points$; $V = set \ of \ right \ points$; $U \cap V = set \ of \ central \ points$.

In particular, if $U \cap V = \phi$, the space is called disjoint, and otherwise it is called joint. Unless otherwise stated, we shall work with joint spaces.
b) A sequence (μ_n) on the set U is called a left sequence, and a sequence (γ_n) on V is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.
c) A sequence (μ_n) is said to be convergent to a point μ if and only if (μ_n) is a left sequence, $\lim_{n \to +\infty} d(\mu_n, \mu) = 0 \ and \ \mu \in V \ or \ (\mu_n)$ is a right sequence. $\lim_{n \to +\infty} d(\mu_n, \mu) = 0 \ and \ \mu \in U$.
d) A bisequence (μ_n, γ_n) on (U,V, d) is a sequence on the set $U \times V$. Furthermore, if the sequences $(\mu_n) \ and \ (\gamma_n)$ are convergent, then the bisequence (μ_n, γ_n) is said to be convergent. In addition, if $(\mu_n) \ and \ (\gamma_n)$ converges to a common point $t \in U \cap V$, then (μ_n, γ_n) is called biconvergent.
e) A bisequence $(\mu_n, \gamma_n)$is a Cauchy bisequence if $\lim_{n \to +\infty} d(\mu_n, \gamma_n) = 0$.

Remark 1.4: In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

Definition 1.5: A bipolar metric space is called complete if every Cauchy bisequence is convergent, hence biconvergent.

Definition 1.6: A covariant or a contravariant map W from the bipolar metric space (U_1, V_1, d_1) to the bipolar metric space $(U_2, V_2, d_2)$is continuous, if and only if $(u_n) \to v$ on $(U_1, V_1, d_1) implies \ W(u_n) \to W(v)$ on (U_2, V_2, d_2).

Definition 1.7 (13): Let $U \neq \phi$ and $d: U \times U \to [0, +\infty)$ is said to be a b-metric if it satisfies the following three conditions

i) $d(\mu, \gamma) = 0 \ if \ and \ only \ if \ \mu = \gamma \ \ \forall \mu, \gamma \in U$
ii) $d(\mu, \gamma) = d(\gamma, \mu) \ for \ any \ \ \mu, \gamma \in U$
iii) $d(\mu, \gamma) \leq \alpha[d(\mu, \gamma) + d(\mu, \gamma)] for \ all \ \mu, \gamma \in U$, $\alpha \geq 1$.

The pair (U, d) is called a b-metric space.

Definition 1.8: Let U be a non-empty set define a mapping $d: U \times U \to [0, +\infty)$ is said to be an extended b-metric if it satisfies the following three conditions:

i) $d_\psi(\mu, \gamma) = 0 \ \ if \ and \ only \ if \ \mu = \gamma \ for \ any \ \mu, \gamma \in U$;
ii) $d_\psi(\mu, \gamma) = d_\psi(\gamma, \mu) \ for \ any \ \ \mu, \gamma \in U$
iii) $d_\psi(\mu, \gamma) \leq \psi(\mu, \gamma) \ (d_\psi(\mu, w) + d_\psi(w, \gamma)) \ for \ all \ u, v, w \in positive \ function \ from \ U \times U \ to \ [0, +\infty)$.

The pair (U, d_ψ) is called an extended b-metric space.

Now we have the following fixed- point theorems on metric spaces.

Theorem 1.9 (Banach's Contraction):

Let (U,d) be complete metric space and $T: U \to U$ satisfying $d(T\mu,T\gamma) \leq \mu_1 d(\mu,\gamma)$ for all $\mu,\gamma \in U$, with $0 \leq \mu_1 < 1$. Then T has a unique fixed point.

Theorem 1.10 (Bannan's Contraction):

Let (U,d) be complete metric space and $T: U \to U$ satisfying $d(T\mu,T\gamma) \leq \mu_1[d(\mu,T\mu)+d(\gamma,d\gamma)]$ for all $\mu,\gamma \in U$, with $0 \leq \mu_1 < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 1.11 (Khan's Contraction):

Let (U,d) be a complete metric space and $T: U \to U$ satisfying $d(T\gamma,T\mu) \leq \mu_1 \frac{d(\mu,T\mu)\ d(\mu,d\gamma)+d(T\gamma,\gamma)d(\gamma,T\mu)}{d(\mu,T\gamma)+d(\gamma,T\mu)}$ for all $\mu,\gamma \in U$, with $0 \leq \mu_1 < 1$. Then T has a unique fixed point.

Theorem 1.12 (Jaggi's Contraction):

Let T be a continuous self map defined on a complete metric space (U,d). Support that T satisfies the following contractive condition $d(T\gamma,T\mu) \leq \mu_1 \frac{d(\mu,T\mu)\ d(T\gamma,\gamma)}{d(\mu,\gamma)} + \mu_2 d(\mu,\gamma)$ for all $\mu,\gamma \in U$, $\mu \neq \gamma$, and for some $\mu_1,\mu_2 \in [0,1)$ with $\mu_1+\mu_2 < 1$. Then T has a unique fixed point in U.

MAIN RESULTS

Extended Bipolar b – Metric Space:

Let U and V be two Nonempty sets and $d_\psi: U \times V \longrightarrow R^+$be a function, (where R^+ denote the set of non-negative real numbers) satisfying the following properties:

i) $d_\psi(u,v) = 0\ iff\ u = v\ for\ any\ (u,v) \in U \times V$

ii) $d_\psi(u,v) = d_\psi(v,u)\ for\ any\ (u,v) \in U \times V$

iii) $d_\psi(u_1,v_2) \leq \psi(u_1,v_2)\big[d_\psi(u_1,v_1) + d_\psi(u_2,v_1) + d_\psi(u_2,v_2)\big]$

for all $u_1,u_2 \in U$ *and* $v_1,v_2 \in V$ where ψ is a positive function from $U \times V$ to $[1,\infty)$.

The pair (u,v,d_ψ) is called an extended bipolar b-metric space.

Now, we establish fixed-point results for mapping satisfying the generalized rational type contractive conditions which extend theorems in extended bipolar b metric spaces.

Theorem 2.1:

Let (u,v,d_ψ) be a complete extended bipolar b metric space and let $f,g: (u,v,d_\psi) \rightleftarrows (u,v,d_\psi)$ be contravariant mapping satisfying

$$d_\psi(fu,gv) \leq \beta_1 d_\psi(u,v) + \beta_2 \frac{d_\psi(u,fu)d_\psi(v,fu)+d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)} \quad \rightarrow (1)$$

where $d_\psi(u,gv)+d_\psi(v,fu) \neq 0$, for all $(u,v) \in U \times V$ with $u \neq v$ *and* $0 \leq \beta_1+\beta_2 < 1$, $\lim_{n\to\infty} \psi(u_n,u_{n+m}) < \frac{1}{\beta} = \frac{\beta_1}{1-\beta_2}$ *where* $\beta_1,\beta_2 \geq 0$.Then $f,g: U \cup V \to U \cup V$ have a unique common fixed point provided that f *and* g are continuous in (U,V).

Proof:

Let $u_0 \in U$ *and* $v_0 \in V$ *for* $n \in N \cup \{0\}$, we use the following one of the iterative methods to define sequences (u_n) *and* (v_n):

$fu_{2n} = v_{2n}$; $gu_{2n+1} = v_{2n+1;}$ $fv_{2n} = u_{2n+1}$; $gv_{2n+1} = u_{2n+2;}$

$$d_\psi(fu,gv) \leq \beta_1 d_\psi(u,v) + \beta_2 \frac{d_\psi(u,fu)d_\psi(v,fu) + d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

$u = v_{2n}$; $v = u_{2n+1}$

Now from equation (1), we get

$$d_\psi(u_{2n+1},v_{2n+1}) = d_\psi(fv_{2n},gu_{2n+1})$$

$$\leq \beta_1 d_\psi(v_{2n},u_{2n+1}) + \beta_2 \frac{d_\psi(v_{2n},fv_{2n})d_\psi(u_{2n+1},fv_{2n}) + d_\psi(u_{2n+1},gu_{2n+1})d_\psi(v_{2n},gu_{2n+1})}{d_\psi(v_{2n},gu_{2n+1}) + d_\psi(u_{2n+1},fv_{2n})}$$

$$\leq \beta_1 d_\psi(u_{2n+1},v_{2n}) + \beta_2 \frac{d_\psi(u_{2n+1},v_{2n})d_\psi(u_{2n+1},u_{2n+1}) + d_\psi(u_{2n+1},v_{2n+1})d_\psi(v_{2n},v_{2n+1})}{d_\psi(v_{2n},v_{2n+1}) + d_\psi(u_{2n+1},u_{2n+1})}$$

$$d_\psi(u_{2n+1},v_{2n+1}) - \beta_2 d_\psi(u_{2n+1},v_{2n+1}) \leq \beta_1 d_\psi(u_{2n+1},v_{2n})$$

$$d_\psi(u_{2n+1},v_{2n+1}) \leq \frac{\beta_1}{1-\beta_2} d_\psi(u_{2n+1},v_{2n}) \quad \rightarrow (2)$$

We obtain also

$$d_\psi(u_{2n+1},v_{2n}) = d_\psi(fv_{2n},fu_{2n})$$

$$\leq \beta_1 d_\psi(v_{2n},u_{2n}) + \beta_2 \frac{d_\psi(v_{2n},fv_{2n}).d_\psi(u_{2n},fv_{2n})+d_\psi(u_{2n},gu_{2n})d_\psi(v_{2n},gu_{2n})}{d_\psi(v_{2n},gu_{2n})+d_\psi(u_{2n},fv_{2n})}$$

$$= \beta_1 d_\psi(v_{2n},u_{2n}) + \beta_2 \frac{d_\psi(v_{2n},u_{2n+1}).d_\psi(u_{2n},u_{2n+1})+d_\psi(u_{2n},v_{2n})d_\psi(v_{2n},v_{2n})}{d_\psi(v_{2n},v_{2n})+d_\psi(u_{2n},u_{2n+1})}$$

$$d_\psi(u_{2n+1},v_{2n}) \leq \beta_1 d_\psi(v_{2n},u_{2n}) + \beta_2 d_\psi(v_{2n},u_{2n+1})$$

$$d_\psi(u_{2n+1},v_{2n}) - \beta_2 d_\psi(v_{2n},u_{2n+1}) \leq \beta_1 d_\psi(v_{2n},u_{2n})$$

$$d_\psi(u_{2n+1},v_{2n}) \leq \frac{\beta_1}{1-\beta_2} d_\psi(v_{2n},u_{2n}) \quad \rightarrow(3)$$

Take $\beta = \frac{\beta_1}{1-\beta_2}$ and then $\beta \in [0,1)$ because $\beta_1 + \beta_2 \in [0,1)$

Hence from equations (2) and (3), we get

$d_\psi(u_{2n+1},v_{2n+1}) \leq \beta^{4n+2} d_\psi(u_0,v_0)$ and

$$d_\psi(u_{2n+1},v_{2n}) \leq \beta^{4n+1} d_\psi(u_0,v_0) \quad \rightarrow(4)$$

Now, we can get that for any $n \in N$,

$$d_\psi(u_{n+1},v_{n+1}) \leq \beta^{2n+2} d_\psi(u_0,v_0)$$

$$d_\psi(u_{n+1},v_n) \leq \beta^{2n+1} d_\psi(u_0,v_0)$$

and

$$d_\psi(u_n,v_n) \leq \beta^{2n} d_\psi(u_0,v_0)$$

Further for all $m,n \in N$, we consider the following cases.

Case 1: *If $m > n$, we have*

$$d_\psi(u_n,v_m) \leq \psi(u_n,v_m)[d_\psi(u_n,v_n) + d_\psi(u_{n+1},v_n) + d_\psi(u_{n+1},v_m)]$$

$$\leq \psi(u_n,v_m)[\beta^{2n} d_\psi(u_0,v_0) + \beta^{2n+1} d_\psi(u_0,v_0) + d_\psi(u_{n+1},v_m)]$$

$$\leq \psi(u_n,v_m)[(\beta^{2n}+\beta^{2n+1}) d_\psi(u_0,v_0) + d_\psi(u_{n+1},v_{n+1}) + d_\psi(u_{n+2},v_{n+1}) + d_\psi(u_{n+2},v_m)]$$

$$\leq \psi(u_n,v_m)[(\beta^{2n}+\beta^{2n+1}) d_\psi(u_0,v_0) + \beta^{2n+2} d_\psi(u_0,v_0) + \beta^{2n+3} d_\psi(u_0,v_0) + d_\psi(u_{n+2},v_m)]$$

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$$\leq \psi(u_n,v_m)[1+\beta+\beta^2+\cdots\ldots\ldots\ldots\ldots\ldots+\beta^{2(m-n)} d_\psi(u_0,v_0)]$$

$$\leq \psi(u_n,v_m)\left[\frac{1-\beta^{2(m-n)+1}}{1-\beta}\right] d_\psi(u_0,v_0)$$

Since $\beta < 1$, $\lim_{m,n\to\infty} d_\psi(u_n,v_m) = 0$

Case 2: *If $m < n$, we have*

$$d_\psi(u_n,v_m) \leq \psi(u_n,v_m)[d_\psi(u_{m+1},v_m) + d_\psi(u_{m+1},v_m) + d_\psi(u_{m+1},v_{m+1}) + d_\psi(u_n,v_{m+1})]$$

$$\leq \psi(u_n,v_m)[\beta^{2m+1} d_\psi(u_0,v_0) + \beta^{2m+2} d_\psi(u_0,v_0) + d_\psi(u_n,v_{m+1})]$$

$$\leq [(\beta^{2m+1}+\beta^{2m+2}) d_\psi(u_0,v_0) + d_\psi(u_{m+2},v_{m+1}) + d_\psi(u_{m+2},v_{m+2}) + d_\psi(u_n,v_{m+2})]$$

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$$\leq \beta\left[1+\beta+\beta+\cdots\ldots\ldots\ldots\ldots\ldots+\beta^{2(n-m-1)}d_\psi(u_0,v_0)\right]$$

$$\leq \beta^{2m+1}\left[\frac{1-\beta^{2(n-m-1)}}{1-\beta}\right]d_\psi(u_0,v_0)$$

Again Since $\beta<1$, $\lim_{m,n\to+\infty} d_\psi(u_n,v_m)=0$

This means that $d_\psi(u_n,v_m)$can be made arbitrarily small by large m and n , and hence $(u_n,v_m)$is a Cauchy bisequence in (U,V). By the completeness of (U,V,d_ψ). , the bisequence (u_n,v_n)binconverges to some $u^*\in U\cap V$ such that $\lim_{n\to+\infty} u_n=\lim_{n\to+\infty} v_n=u^*$. Also, $fu_{2n}=v_{2n}\to u^*\in U\cap V$implies that fu_{2n}has a unique limit u^* and$u_n\to u^*$ implies that $u_{2n}\to u^*$. Now, the continuity of f implies that $fu_{2n}\to u^*$. Therefore $fu^*=u^*$.

Similarly, $gv_{2n+1}=u_{2n+2}\to u^*\in U\cap V$implies that gv_{2n+1}has a unique limit $u^*$and $v_n\to u^*$implies that $v_{2n+1}\to u^*$. Now, the continuity of g implies that $g(v_{2n+1})\to gu^*$. Therefore, $gu^*=u^*$. Thus,

$fu^*=gu^*=u^*$, i.e., f and g have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. If $v^*\in U\cap V$is another common fixed point of f and g ,that is, $fv^*=gv^*=v^*\in U\cap V$ then we get

$$d_\psi(v^*,u^*)=d_\psi(fv^*,gu^*)$$

$$\leq \beta_1 d_\psi(u^*,v^*)+\beta_2\frac{d_\psi(u^*,fu^*)d_\psi(v^*,fu^*)+d_\psi(v^*,gv^*)d_\psi(u^*,gv^*)}{d_\psi(u^*,gv^*)+d_\psi(v^*,fu^*)}$$

$$=\beta_1 d_\psi(u^*,v^*)+\beta_2\frac{d_\psi(u^*,u^*)d_\psi(v^*,u^*)+d_\psi(v^*,v^*)d_\psi(u^*,v^*)}{d_\psi(u^*,v^*)+d_\psi(v^*,u^*)}$$

Therefore, $d_\psi(v^*,u^*)\leq\beta_1 d_\psi(u^*,v^*)$, which is a contradiction, and hence $u^*=v^*$. This completes the theorem.

Theorem 2.2:

Let (u,v,d_ψ) be a complete extended bipolar b metric space and let $f,g:(u,v,d_\psi)\rightleftarrows(u,v,d_\psi)$ be contravariant mapping satisfying

$$d_\psi(fu,gv)\leq\beta_1 d_\psi(u,v)+\beta_2\frac{d_\psi(u,fu)d_\psi(v,fu)+d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}+\beta_3\frac{d_\psi(v,fu)d_\psi(u,gv)+d_\psi(v,gv)d_\psi(v,gu)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

$$\rightarrow(5)$$

for all $(u,v)\in U\times V$, with $u\neq v$ *and* $0\leq\beta_1+\beta_2+\beta_3<1$ where $d_\psi(u,gv)+d_\psi(v,fu)\neq 0$,

$\beta_1,\beta_2,\beta_3\in[0,1)$*and* $\lim_{n\to\infty}\psi(u_n,u_{n+m})<\frac{1}{\gamma}=\frac{\beta_1}{1-\beta_2}$. Then $f,g:U\cup V\to U\cup V$ have a unique common fixed point provided that f *and* g are continuous in (U,V).

Proof:

Let $u_0\in U$ *and* $v_0\in V$ *for* $n\in N\cup\{0\}$, we use the following one of the iterative methods to define sequences (u_n) *and* (v_n):

$$fu_{2n}=v_{2n}\,;\ gu_{2n+1}=v_{2n+1;}\ \ fv_{2n}=u_{2n+1};\ \ gv_{2n+1}=u_{2n+2;}$$

$$d_\psi(fu,gv)\leq\beta_1 d_\psi(u,v)+\beta_2\frac{d_\psi(u,fu)d_\psi(v,fu)+d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

$$+\beta_3\frac{d_\psi(v,fu)d_\psi(u,gv)+d_\psi(v,gv)d_\psi(v,gu)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

$$u=v_{2n};\quad v=u_{2n+1}$$

Now from equation (5), we get

$d_\psi(u_{2n+1},v_{2n+1})=d_\psi(fv_{2n},gu_{2n+1})$

$$\leq \beta_1 d_\psi(v_{2n}, u_{2n+1}) + \beta_2 \frac{d_\psi(v_{2n}, fv_{2n}) d_\psi(u_{2n+1}, fv_{2n}) + d_\psi(u_{2n+1}, gu_{2n+1}) d_\psi(v_{2n}, gu_{2n+1})}{d_\psi(v_{2n}, gu_{2n+1}) + d_\psi(u_{2n+1}, fv_{2n})}$$
$$+ \beta_3 \frac{d_\psi(u_{2n+1}, fv_{2n}) d_\psi(v_{2n}, gu_{2n+1}) + d_\psi(u_{2n+1}, gu_{2n+1}) d_\psi(u_{2n+1}, gv_{2n})}{d_\psi(v_{2n}, gu_{2n+1}) + d_\psi(u_{2n+1}, fv_{2n})}$$

$$\leq \beta_1 d_\psi(u_{2n+1}, v_{2n}) + \beta_2 \frac{d_\psi(u_{2n+1}, v_{2n}) d_\psi(u_{2n+1}, u_{2n+1}) + d_\psi(u_{2n+1}, v_{2n+1}) d_\psi(v_{2n}, v_{2n+1})}{d_\psi(v_{2n}, v_{2n+1}) + d_\psi(u_{2n+1}, u_{2n+1})}$$
$$+ \beta_3 \frac{d_\psi(u_{2n+1}, u_{2n+1}) d_\psi(v_{2n}, v_{2n+1}) + d_\psi(u_{2n+1}, v_{2n+1}) d_\psi(u_{2n+1}, u_{2n+1})}{d_\psi(v_{2n}, v_{2n+1}) + d_\psi(u_{2n+1}, u_{2n+1})}$$

$$d_\psi(u_{2n+1}, v_{2n+1}) \leq \beta_1 d_\psi(u_{2n+1}, v_{2n}) + \beta_2 d_\psi(u_{2n+1}, v_{2n+1})$$
$$d_\psi(u_{2n+1}, v_{2n+1}) - \beta_2 d_\psi(u_{2n+1}, v_{2n+1}) \leq \beta_1 d_\psi(u_{2n+1}, v_{2n})$$
$$d_\psi(u_{2n+1}, v_{2n+1}) \leq \frac{\beta_1}{1-\beta_2} d_\psi(u_{2n+1}, v_{2n}) \quad \rightarrow (6)$$

Take $\beta = \frac{\beta_1}{1-\beta_2}$ and then $\beta \in [0,1)$ because $\beta_1 + \beta_2 \in [0,1)$

Hence from equations (5) and (6), we get

$$d_\psi(u_{2n+1}, v_{2n+1}) \leq \beta^{4n+2} d_\psi(u_0, v_0) \text{ and}$$
$$d_\psi(u_{2n+1}, v_{2n}) \leq \beta^{4n+1} d_\psi(u_0, v_0) \quad \rightarrow (7)$$

Now, we can get that for any $n \in N$,

$$d_\psi(u_{n+1}, v_{n+1}) \leq \beta^{2n+2} d_\psi(u_0, v_0)$$
$$d_\psi(u_{n+1}, v_n) \leq \beta^{2n+1} d_\psi(u_0, v_0)$$

and

$$d_\psi(u_n, v_n) \leq \beta^{2n} d_\psi(u_0, v_0)$$

Further for all $m, n \in N$, we consider the following cases.

Case 1: *If $m > n$, we have*

$$d_\psi(u_n, v_m) \leq \psi(u_n, v_m)[d_\psi(u_n, v_n) + d_\psi(u_{n+1}, v_n) + d_\psi(u_{n+1}, v_m)]$$
$$\leq \psi(u_n, v_m)[\beta^{2n} d_\psi(u_0, v_0) + \beta^{2n+1} d_\psi(u_0, v_0) + d_\psi(u_{n+1}, v_m)]$$
$$\leq \psi(u_n, v_m)[(\beta^{2n} + \beta^{2n+1}) d_\psi(u_0, v_0) + d_\psi(u_{n+1}, v_{n+1}) + d_\psi(u_{n+2}, v_{n+1}) + d_\psi(u_{n+2}, v_m)]$$
$$\leq \psi(u_n, v_m)[(\beta^{2n} + \beta^{2n+1}) d_\psi(u_0, v_0) + \beta^{2n+2} d_\psi(u_0, v_0) + \beta^{2n+3} d_\psi(u_0, v_0) + d_\psi(u_{n+2}, v_m)]$$

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$$\leq \psi(u_n, v_m)[1 + \beta + \beta^2 + \cdots \ldots \ldots \ldots \ldots \ldots + \beta^{2(m-n)} d_\psi(u_0, v_0)]$$

$$\leq \psi(u_n, v_m) \left[\frac{1 - \beta^{2(m-n)+1}}{1-\beta}\right] d_\psi(u_0, v_0)$$

Since$\beta < 1$, $\lim_{m,n \to \infty} d_\psi(u_n, v_m) = 0$

Case 2: *If $m < n$, we have*

$$d_\psi(u_n, v_m) \leq \psi(u_n, v_m)[d_\psi(u_{m+1}, v_m) + d_\psi(u_{m+1}, v_m) + d_\psi(u_{m+1}, v_{m+1}) + d_\psi(u_n, v_{m+1})]$$
$$\leq \psi(u_n, v_m)[\beta^{2m+1} d_\psi(u_0, v_0) + \beta^{2m+2} d_\psi(u_0, v_0) + d_\psi(u_n, v_{m+1})]$$
$$\leq [(\beta^{2m+1} + \beta^{2m+2}) d_\psi(u_0, v_0) + d_\psi(u_{m+2}, v_{m+1}) + d_\psi(u_{m+2}, v_{m+2}) + d_\psi(u_n, v_{m+2})]$$

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$$\leq \beta\left[1+\beta+\beta+\cdots\ldots\ldots\ldots\ldots\ldots+\beta^{2(n-m-1)}d_\psi(u_0,v_0)\right]$$

$$\leq \beta^{2m+1}\left[\frac{1-\beta^{2(n-m-1)}}{1-\beta}\right]d_\psi(u_0,v_0)$$

Again Since $\beta<1$, $\lim_{m,n\to+\infty} d_\psi(u_n,v_m)=0$

This means that $d_\psi(u_n,v_m)$can be made arbitrarily small by large m and n , and hence $(u_n,v_m)$is a Cauchy bisequence in (U,V). By the completeness of (U,V,d_ψ). , the bisequence (u_n,v_n)binconverges to some $u^*\in U\cap V$ such that $\lim_{n\to+\infty} u_n=\lim_{n\to+\infty} v_n=u^*$. Also, $fu_{2n}=v_{2n}\to u^*\in U\cap V$implies that fu_{2n}has a unique limit u^* and$u_n\to u^*$ implies that $u_{2n}\to u^*$. Now, the continuity of f implies that $fu_{2n}\to u^*$. Therefore $fu^*=u^*$.

Similarly, $gv_{2n+1}=u_{2n+2}\to u^*\in U\cap V$implies that gv_{2n+1}has a unique limit $u^*$and $v_n\to u^*$implies that $v_{2n+1}\to u^*$. Now, the continuity of g implies that $g(v_{2n+1})\to gu^*$. Therefore, $gu^*=u^*$. Thus,

$fu^*=gu^*=u^*$, i.e., f and g have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. If $v^*\in U\cap V$is another common fixed point of f and g that is, $fv^*=gv^*=v^*\in U\cap V$ then we get

$$d_\psi(v^*,u^*)=d_\psi(fv^*,gu^*)$$

$$\leq \beta_1 d_\psi(u^*,v^*)+\beta_2\frac{d_\psi(u^*,fu^*)d_\psi(v^*,fu^*)+d_\psi(v^*,gv^*)d_\psi(u^*,gv^*)}{d_\psi(u^*,gv^*)+d_\psi(v^*,fu^*)}+\beta_3\frac{d_\psi(v^*,fu^*)d_\psi(u^*,gv^*)+d_\psi(v^*,gv^*)d_\psi(v^*,gu^*)}{d_\psi(u^*,gv^*)+d_\psi(v^*,fu^*)}$$

$$=\beta_1 d_\psi(u^*,v^*)+\beta_2\frac{d_\psi(u^*,u^*)d_\psi(v^*,u^*)+d_\psi(v^*,v^*)d_\psi(u^*,v^*)}{d_\psi(u^*,v^*)+d_\psi(v^*,u^*)}+\beta_3\frac{d_\psi(v^*,u^*)d_\psi(u^*,v^*)+d_\psi(v^*,v^*)d_\psi(v^*,u^*)}{d_\psi(u^*,v^*)+d_\psi(v^*,u^*)}$$

Therefore, $d_\psi(v^*,u^*)\leq\beta_1 d_\psi(u^*,v^*)$, which is a contradiction, and hence $u^*=v^*$. This completes the theorem.

Corollary 2.3:

Let (u,v,d_ψ) be a complete extended bipolar b metric space and let $f,g:(u,v,d_\psi)\rightleftarrows(u,v,d_\psi)$ be contravariant mapping satisfying

$$d_\psi(fu,gv)\leq\beta_1 d_\psi(u,v)+\beta_2\frac{d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

where $d_\psi(u,gv)+d_\psi(v,fu)\neq 0$, for all $(u,v)\in U\times V$ with $u\neq v$ *and* $0\leq\beta_1+\beta_2<1$, $\lim_{n\to\infty}\psi(u_n,u_{n+m})<\frac{1}{\beta}=\frac{\beta_1}{1-\beta_2}$ *where* $\beta_1,\beta_2\geq 0$.Then $f,g:U\cup V\to U\cup V$ have a unique common fixed point provided that f *and* g are continuous in $(U,V$

Corollary 2.4:

Let (u,v,d_ψ) be a complete extended bipolar b metric space and let $f,g:(u,v,d_\psi)\rightleftarrows(u,v,d_\psi)$ be contravariant mapping satisfying

$$d_\psi(fu,gv)\leq\beta_1 d_\psi(u,v)+\beta_2\frac{d_\psi(v,gv)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}+\beta_3\frac{d_\psi(v,fu)d_\psi(u,gv)}{d_\psi(u,gv)+d_\psi(v,fu)}$$

for all $(u,v)\in U\times V$, with $u\neq v$ *and* $0\leq\beta_1+\beta_2+\beta_3<1$ where $d_\psi(u,gv)+d_\psi(v,fu)\neq 0$,

$\beta_1,\beta_2,\beta_3\in[0,1)$*and* $\lim_{n\to\infty}\psi(u_n,u_{n+m})<\frac{1}{\gamma}=\frac{\beta_1}{1-\beta_2}$. Then $f,g:U\cup V\to U\cup V$ have a unique common fixed point provided that f *and* g are continuous in (U,V).

APPLICATION TO HOMOTOPY THEORY

Let X be a complete extended bipolar b-metric space and let $F,G:X\to X$ be two continuous contravariant mappings satisfying the contraction condition given in Theorem 2.1

Let $H: X \times [0,1] \to X$ be a homotopy defined by

$H(x,t) = (1-t)F(x) + tG(x), \forall\ x \in X,\ t \in [0,1]$

Then there exists a unique element $x^* \in X$ such that

$$H(x^*,t) = x^*\ \ for\ all\ t \in [0,1].$$

Proof:

Since F and G satisfy the contraction condition and are continuous, there exists a unique common fixed point

$$x^* \in X\ such\ that$$

$$F(x^*) = x^*, G(x^*) = x^*$$

Define the homotopy

$H(x,t) = (1-t)F(x) + tG(x), t \in [0,1]$

Since F and G are continuous, H is also continuous in both variables.

Substitute $x = x^*\ in\ H(x,t)$

$$H(x^*,t) = (1-t)F(x^*) + tG(x^*)$$

$$F(x^*) = x^*, G(x^*) = x^*$$

Weget

$$H(x^*,t) = (1-t)x^* + tx^*$$

$$= x^*[(1-t)+t]$$

$$= x^*$$

Thus

$$H(x^*,t) = x^*\ for\ every\ \ t \in [0,1]$$

x^* is a fixed point for every mapping in the homotopy family $H(.,t)$.

Hence a solution exists for all $t \in [0,1]$, i.e., the solution persists under continuous deformation from F to G.

Assume that there exists another point $y^* \in X$ such that

$$H(y^*,t) = y^*\ for\ all\ t \in [0,1]$$

$$put\ t = 0, \quad H(y^*,0) = F(y^*) = y^*$$

$$put\ t = 1, \quad H(y^*,1) = G(y^*) = y^*$$

Thus y^* is also a common fixed point of F and G.

By theorem 2.1 we get $y^* = x^*$

Therefore, there exists a unique point $x^* \in X$ such that

$$H(x^*,t) = x^*\ \ for\ all\ t \in [0,1]$$

DISCUSSION

The two main theorems establish that the rational-type contractive conditions long studied in ordinary and extended b-metric spaces carry over to the bipolar setting once the underlying space is equipped with a point-dependent control function ψ. Theorem 2.1 extends the Khan-type rational contraction, while Theorem 2.2 adds a third rational term structurally related to the Jaggi contraction, and the two results together show that the extension is not tied to a single contraction form: the same bisequence and telescoping argument goes through whether the contractive bound has two rational terms or three, provided the combined constants stay below 1 and the limiting condition on ψ holds along the iterative bisequence.

A point worth drawing out is the role of the contravariant structure. Because F and G map $U \cup V$ into itself but reverse sides ($F, G: (U,V) \rightleftarrows (U,V)$), the bisequence construction alternates between the two sets at every step, which is what forces Case 1 and Case 2 of the Cauchy argument to be treated separately rather than as a single symmetric case, unlike the classical single-metric setting. This is the structural reason the proofs run longer than their ordinary b-metric analogues, and it is also what makes the bipolar setting genuinely useful for two-sided problems rather than a relabeling of the one-sided theory.

The homotopy application illustrates a further point, because the fixed point of the contraction family $H(x,t) = (1-t)F(x) + tG(x)$ is forced to coincide with the common fixed point of F and G at both endpoints $t = 0$ and $t = 1$, uniqueness in Theorem 2.1 is doing real work, it is what rules out the family having a different fixed point at some intermediate t. This connects the fixed-point result to a genuinely topological statement about deformation rather than being an add-on example.

A limitation of the present work is that both theorems require continuity of F and G, whether the results extend to non-continuous contravariant mappings, as has been done in some extended b-metric settings, is left open. Likewise, the paper does not address well-posedness or Ulam–Hyers stability of the fixed point, both of which are natural next questions once existence and uniqueness are established in this setting.

CONCLUSION

In this paper we introduced the notion of an extended bipolar b-metric space, obtained by equipping a bipolar b-metric with a point-dependent control function ψ in place of a fixed constant, in the same spirit as Kamran et al.'s extension of ordinary b-metric spaces. Within this setting we proved two common fixed-point theorems for contravariant mappings satisfying generalized rational-type contractive conditions the first extending a Khan-type bound, the second adding a Jaggi-type rational term together with two corollaries obtained as special cases. Both

theorems were established via a bisequence argument, showing that the associated bisequence is Cauchy and biconverges to a common fixed point of the two mappings, with uniqueness following from a direct contradiction argument.

We further applied these results to homotopy theory, showing that the common fixed point of two continuous contravariant mappings F and G persists throughout the entire homotopy family $H(\cdot, t) = (1 - t)F + tG \; for \; t \in [0, 1]$, and that this persisting fixed point is unique. This gives a concrete link between the abstract fixed-point machinery developed here and questions of continuous deformation in topology and nonlinear analysis.

The results extend and unify several existing lines of work on extended b-metric spaces, on bipolar b-metric spaces, and on rational-type contractions within a single framework, and the two corollaries show that simpler, previously studied contraction conditions are recovered as special cases rather than requiring separate proofs. Directions for future work include relaxing the continuity requirement on F and G, examining well-posedness and Ulam–Hyers stability of the fixed point in this setting, and extending the framework to multivalued or fuzzy contravariant mappings, following the pattern set by recent generalizations of the extended b-metric space.

ACKNOWLEDGEMENT

None.

Conflict of interest: None

Source of support: None

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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