The Scientific Temper
AUGUST 2026 Vol. 17 (8): 6733-6739
ISSN: 0976-8653, E-ISSN: 2231-6396
DOI: 10.58414/SCIENTIFICTEMPER.2026.17.8.2516
A WEB OF SCIENCE JOURNAL

RESEARCH ARTICLE

A Non-Iterative Set-Theoretic Approach to Fixed Point Theorems in Multiplicative Banach Spaces

IMTIYAZ MD[1*], B. KRISHNA REDDY[2]

Research Scholar, Department of Mathematics, University College of Science, Osmania University, Hyderabad, India, Email: imtiyazmd83@gmail.com. https://orcid.org/0009-0001-1545-1950

[2]Department of Mathematics, University College of Science, Osmania University, Hyderabad, India, Email.bkreddy345@gmail.com

***Corresponding author: imtiyazmd83@gmail.com**

ABSTRACT

This paper introduces an alternative approach to proving fixed point theorems, by passing the conventional iterative method. Instead, we employ sets and intersections, inspired by Boyd and Wong's concept. Our focus is on multiplicative normed spaces, examining three types of contractions: Banach, Kannan, and Chatterjea. The key technique involves considering sets of "almost fixed points," which are points located within a certain distance from their image. As this distance approaches 1, these sets diminish and converge to a single point. Cantor's intersection theorem then directly yields the fixed point. We present new formulas that describe the rate at which these sets contract for each type of contraction. It is also noted that multiplicative normed spaces are essentially regular normed spaces under a different appearance, so the existence results are not novel but merely rephrased. What is novel are the speed estimates. Examples, including an application to a Volterra integral equation, demonstrate the findings.

Keywords: *Multiplicative normed space, Cantor intersection theorem, fixed point, contraction, Baire category theorem, diameter estimates, non-iterative proof.*

AMS Subject Classification (2020): Primary 47H10; Secondary 54H25, 46B99.

INTRODUCTION

The construction of successive iterates is a standard method for establishing fixed points of contraction mappings. In this approach, an initial element is repeatedly transformed by the mapping and the resulting sequence is analyzed to obtain convergence to a fixed point. Banach-type, Kannan-type, and Chatterjea type contractions are among the important contraction classes studied within this framework (Chatterjea ,1972).

A different strategy avoids the explicit construction of an iterative orbit. Boyd and Wong (1968) demonstrated such an approach by using Cantor's Intersection Theorem in their treatment of the Banach contraction principle. Related set-theoretic arguments were subsequently considered by Kannan (1969) and Lahiri et al. (2011) in other fixed point settings. The central idea is to study suitable nested approximate fixed point sets and use their shrinking geometric properties to obtain the desired fixed point.

Received: 8th August, 2026 **Accepted:** 12th August, 2026 **Published:** 28th August, 2026

Multiplicative Normed Linear Spaces (MNLS), introduced by Chugh et al. (2017), provide a framework in which norm-related concepts can be formulated multiplicatively. Subsequent work, including the results of Özavşar and Çevikel (2017), has established fixed point results in multiplicative settings through iterative techniques. This naturally raises the question of whether the set-theoretic approach based on approximate fixed point sets can also be developed directly in multiplicative Banach spaces.

The present work develops a non-iterative fixed-point framework within multiplicative Banach spaces. First, we establish a multiplicative Cantor intersection principle and a corresponding Baire category result. We then introduce multiplicative approximate fixed-point sets and determine explicit bounds for their multiplicative diameters under Banach, Kannan, and Chatterjea type contraction conditions. These estimates are used to obtain the existence and uniqueness of fixed points through the Cantor intersection property. Finally, the resulting framework is applied to a nonlinear Volterra integral equation.

PRELIMINARIES

Throughout this paper, S denotes a linear space.

Following Chugh et al. (2017), a multiplicative norm on S is a mapping $\|.\|^*: S \to [1,\infty)$, satisfy (1). $\|u\|^* \geq 1$, with equality holding strictly when $u = 0$.

(2). $\|\alpha u\|^* = (\|u\|^*)^{|\alpha|}$ (Power homogeneity)

(3). $\|u+v\|^* \leq \|u\|^*.\|v\|^*$ (Multiplicative Sub-additivity).

The pair $(S, \|.\|^*)$ is formally recognized as a Multiplicative Normed Linear Space (MNLS) . The multiplicative topology is generated by the multiplicative open ball $B_\varepsilon(u) = \{ v \in S: \|u-v\|^* < \epsilon \}$, $\epsilon > 1$. equivalently, $u \in \bar{A}$ if and only if there exists a sequence $\{u_n\} \subset A$ with $u_n \to * \ u$.

A sequence $\{u_n\}$ in S is multiplicative converges to $u \in S$ (written $u_n \to * \ u$) if for every $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $\| u_n - u \|^* < \varepsilon$ for all $n \geq N$. Or if $\| u_n - u \|^* \to 1$ as $n \to \infty$.

And it is multiplicative Cauchy, if $\| u_n - u_m \|^* \to 1$ as $n,m \to \infty$; i.e., for every $\epsilon > 1$, there exists N such that $\| u_n - u_m \|^* < \epsilon$ for all $n, m \geq N$. Or if $\| u_n - u_m \|^* \to 1$ as $m,n \to \infty$.

An MNLS S is said to be a Multiplicative Banach space, if every multiplicative Cauchy sequence converges to a point in S. These notions are standard and may be found in Chugh et al. (2017).

For later use, we also recall the notion of multiplicative nowhere dense sets. A subset $A \subseteq S$ is called multiplicatively nowhere dense if the interior of its multiplicative closure is empty.

The following proposition establishes the connection between multiplicative and classical normed spaces and serves as the foundation for the subsequent results.

Proposition 2.1. Let $(S, \|\cdot\|^*)$ be a multiplicative normed linear space. Define $\|u\| = ln\|u\|^*$. Then $\|.\|$ is an ordinary norm on S. Moreover:

1. Multiplicative convergence is equivalent to ordinary norm convergence;
2. Multiplicative Cauchy sequences are precisely the ordinary Cauchy sequences;
3. Multiplicatively closed subsets coincide with closed subsets in $(S, \|.\|)$;
4. $(S, \|\cdot\|^*)$ is multiplicatively complete if and only if $(S, \|.\|)$ is a Banach space.

The above correspondence allows classical topological arguments to be interpreted in the multiplicative setting while preserving the multiplicative formulation adopted throughout this paper. The subsequent sections develop multiplicative versions of Cantor's Intersection Theorem and the Baire Category Theorem that are used to obtain the quantitative diameter estimates for multiplicative contractions.

MULTIPLICATIVE CANTOR AND BAIRE THEOREMS

In this section, we establish multiplicative versions of **Cantor's Intersection Theorem** and the **Baire Category Theorem**, which provide the principal topological tools for the fixed-point results developed later. Although these results correspond to their classical counterparts through Proposition 2.1, direct multiplicative proofs are presented to preserve the multiplicative framework and to facilitate the diameter estimates established in Section 4

Definition 3.1 (Multiplicative Diameter). Let A be a nonempty subset of an MNLS S. The multiplicative diameter of A, denoted by $\delta_m(A)$ and is defined as $\delta_m(A) = Sup\{\| u - v \|^*: u, v \in A\}$. Equivalently $\delta_m(A) = Sup_{u,v \in A} \| u - v \|^*$. Observe that $\delta_m(A) \geq 1$.

Lemma 3.2. For $A \subseteq B \subseteq S$,

1. $A \subseteq B$, then $\delta_m(A) \leq \delta_m(B)$.
2. $\delta_m(A) = 1$ if and only if A consists of exactly one point.
3. A Sequence $\{u_n\}$ is multiplicative Cauchy iff $\delta_m(\{u_n: n \geq N\}) \to 1$ as $n \to \infty$.

4. $\delta_m(\bar{A}) = \delta_m(A)$, where $\bar{A}$ denotes the multiplicative closure of A (i.e., A together with all its multiplicative limit points).

Proof. The assertions follow directly from the definition of multiplicative diameter, the multiplicative norm axioms, and the continuity of the multiplicative norm.

Lemma 3.3

Let $(S, \|\cdot\|^*)$ be an MNLS and let $B_\epsilon(u_0)$ be a multiplicative closed ball with centre u_0 and radius $\epsilon > 1$. Then:

1. $\delta_m(\bar{B}_\epsilon(u_0))) \leq \epsilon^2$.

2. If (ϵ_n) $n \in \mathbb{N}$ is a sequence of real numbers such that $\epsilon_n > 1$ for all n and $\lim_{n\to\infty} \epsilon_n = 1$ (denoted $\epsilon_n \to 1^+$ in $\mathbb{R}$), then $\delta_m(B_{\epsilon_n}(u_0)) \to 1$ (in $\mathbb{R}$).

Proof. Both statements follow immediately from the multiplicative triangle inequality.

Theorem 3.4 (Multiplicative Cantor Intersection Theorem).

Let S be a multiplicative normed linear space (MNLS). Then S is multiplicatively complete (i.e., a multiplicative Banach space) if and only if every decreasing sequence of nonempty multiplicatively closed subsets

$$F_1 \supseteq F_2 \supseteq \cdots$$

Satisfying $\lim_{n\to\infty} \delta_m(F_n) = 1$, has a unique common point.

Proof. Suppose S is multiplicatively complete. Let $\{F_n\}$ satisfy the hypotheses of the theorem. Since each F_n is nonempty. Choose $u_n \in F_n$. Since $u_n, u_m \in F_n$ whenever $\geq n$, $\|u_n - u_m\|^* \leq \delta_m(F_n) \to 1$. Hence $\{u_n\}$ is multiplicatively Cauchy, so $u_n \to* u$ for some $\in S$. Fix $k \in \mathbb{N}$, for every $n \geq k$, the choice of u_n gives $u_n \in F_n \subseteq F_k$ As each F_k is multiplicatively closed and $u_n \to* u$, we obtain $u \in F_k$. Because k is arbitrary, $u \in \bigcap_{n=1}^{\infty} F_n$. Suppose $u, v \in \bigcap_{n=1}^{\infty} F_n$, then for every n, $\| u - v \|^* \leq \delta_m(F_n)$. Let $n \to \infty$. Since $\delta_m(F_n) \to 1$, we get $\| u - v \|^* \leq 1$, Since every multiplicative norm is at least 1 we obtain $\| u - v \|^* = 1$, and hence $u = v$. So, the intersection is a singleton.

Conversely, assume S satisfies the Multiplicative Cantor Intersection Property. Let $\{u_n\}$ be a multiplicative Cauchy sequence. Define $A_n = \{u_n, u_{n+1}, \ldots\}$ and let $F_n = \bar{A}_n$. Then establish, one at a time: $F_{n+1} \subseteq F_n, F_n \neq \emptyset$ and F_n is multiplicatively closed, $\delta_m(A_n) \to 1$. $\delta_m(F_n) = \delta_m(\overline{A_n}) = \delta_m(A_n) \to 1$. By the Multiplicative CIP, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Choose $\in \bigcap_{n=1}^{\infty} F_n$. Let $\epsilon > 1$. Since $\delta_m(F_n) \to 1$, there exists N such that $\delta_m(F_N) < \epsilon$. For every $n \geq N$, $u_n \in F_N$ and also $u \in F_N$. Therefore, $\| u_n - u \|^* \leq \delta_m(F_N) < \epsilon$. Thus $u_n \to* u$. Hence $S$is multiplicatively complete.

Lemma 3.5: A subset $D \subseteq S$ is multiplicatively dense in S (i.e., $\bar{D} = S$) if and only every nonempty multiplicatively open subset of S intersects D.

Proof: This follows directly from the definition of multiplicative closure.

Theorem 3.6 (Multiplicative Baire Category Theorem)

Every multiplicative Banach space is of second multiplicative category; equivalently, it cannot be represented as a countable union of multiplicatively nowhere dense subsets.

Proof. Assume, to the contrary, that $S = \bigcup_{n=1}^{\infty} A_n$, where each A_n is multiplicatively nowhere dense. We construct inductively a decreasing sequence of nonempty multiplicatively closed balls $\{B_n\}$ such that $B_n \cap \overline{A_n} = \emptyset$, for $n \in \mathbb{N}$. Since $\overline{A_1}$ has empty interior, there exists a multiplicatively closed ball $B_1 \subseteq S \setminus \overline{A_1}$. Suppose B_n has been constructed. As $\overline{A_{n+1}}$ also has empty interior . Hence there exists a multiplicatively closed ball $B_{n+1} \subseteq B_n \setminus \overline{A_{n+1}}$. chosen so that its radius $\epsilon_{n+1} > 1$ satisfies $\epsilon_{n+1} \downarrow 1$ (monotonically decreasing sequence converges to 1). Thus $\{B_n\}$ is a decreasing sequence of nonempty multiplicatively closed balls. By Lemma 3.3,$\delta_m(B_n) < \epsilon_n^2 \to 1$. Therefore, Theorem 3.4 yields $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$. Choose $u \in \bigcap_{n=1}^{\infty} B_n$. Since $B_n \cap \overline{A_n} = \emptyset$. we have $u \notin A_n$for every n , ontradicting $S = \bigcup_{n=1}^{\infty} A_n$. Hence S cannot be expressed as a countable union of multiplicatively nowhere dense sets.

MULTIPLICATIVE APPROXIMATE FIXED-POINT SETS AND DIAMETER ESTIMATES

In this section, we introduce multiplicative approximate fixed-point sets associated with multiplicative contractions. We establish their basic properties and derive explicit diameter estimates for Banach, Kannan, and Chatterjea type contractions. These estimates constitute the principal technical contribution of this paper.

Definition 4.1 (Multiplicative contraction).

Let $(S, \|\cdot\|^*)$ be a multiplicative normed linear space. A mapping $T: S \to S$ is called

(1). *Multiplicative Banach-Type Contraction* if there exists a constant $\alpha \in (0,1)$ such that

$\| Tu - Tv \|^* \leq (\| u - v \|^*)^\alpha$, $\forall u, v \in S$.

(2) *Kannan-Type Contraction*, if there exists $\alpha \in (0, \frac{1}{2})$ such that $\| Tu - Tv \|^* \leq (\| u - Tu \|^* . \| v - Tv \|^*)^\alpha$, $\forall u, v \in S$.

(3) *Chatterjea-Type Contraction* if there exists $\alpha \in (0, \frac{1}{2})$ such that $\| Tu - Tv \|^* \leq (\| u - Tv \|^* . \| v - Tu \|^*)^\alpha$, $\forall u, v \in S$.

Lemma 4.2 Every multiplicative Banach-type contraction is multiplicatively continuous.

Proof: Assume $\boldsymbol{u_n} \to_* \boldsymbol{u}$, so $\| \boldsymbol{u_n} - \boldsymbol{u} \|^* \to \mathbf{1}$. By the multiplicative contraction property,

$\| Tu_n - Tu \|^* \leq (\| u_n - u \|^*)^\alpha$. Since, $\| u_n - u \|^* \to 1$, we obtain $\| Tu_n - Tu \|^* \to 1$. Hence $Tu_n \to_* Tu$.

Remark 4.3. Continuity does not generally follow from the Kannan or Chatterjea contraction conditions. Therefore, continuity is assumed in Theorem 5.1 to ensure that the approximate fixed-point sets are multiplicatively closed.

Definition 4.4 (Multiplicative Approximate fixed-point set).

Let $(S, \|\cdot\|^*)$ be an MNLS and $T: S \to S$ a mapping. For any $\epsilon > 1$, define

$A_\epsilon(T) = \{u \in S : \| u - Tu \|^* \leq \epsilon\}$. The set $A_\epsilon(T)$ is called the **multiplicative approximate fixed-point set**.

Lemma 4.5 (Properties of $A_\epsilon(T)$)

Let $T: S \to S$ be a multiplicative continuous. Assume that $A_\epsilon(T) \neq \emptyset$ for every $\epsilon > 1$. Then

(1) $A_\epsilon(T)$ is multiplicative closed

(2) If $1 < \epsilon_1 < \epsilon_2$ then $A_{\epsilon_1}(T) \subseteq A_{\epsilon_2}(T)$

Proof:

(1) *Multiplicative Closedness*. If $u_n \in A_\epsilon(T)$ and $u_n \to_* u$ (i.e. $\| u_n - u \|^* \to 1$), multiplicative continuity of T and the norm gives $\| u - Tu \|^* = \lim_{n\to\infty} \| u_n - Tu_n \|^* \leq \epsilon$. Thus $u \in A_\epsilon(T)$.

(2) *Nesting*. If $1 < \epsilon_1 < \epsilon_2$ and $u \in A_{\epsilon_1}(T)$,then$\| u - Tu \|^* \leq \epsilon_1 < \epsilon_2$, so $u \in A_{\epsilon_2}(T)$.Hence $A_{\epsilon_1}(T) \subseteq A_{\epsilon_2}(T)$.

Theorem 4.6 (Diameter Estimates for Multiplicative Contractions)

Let $T: S \to S$ a mapping satisfying one of the three contraction conditions form the definition 4.1.

For $\epsilon > 1$, let $A_\epsilon(T) = \{u : \| u - Tu \|^* \leq \epsilon\}$. Then the multiplicative diameter of $A_\epsilon(T)$ satisfies

$$\delta_m(A_\epsilon(T)) \leq \begin{cases} \epsilon^{\frac{2}{1-\alpha}}, & (Banach) \\ \epsilon^{2+2\alpha}, & (Kannan) \\ \epsilon^{(\frac{2+2\alpha}{1-2\alpha})}, & (Chaterjia) \end{cases}$$

consequently, if $\epsilon_n \to 1^+$, then $\delta_m(A_\epsilon(T)) \to 1$ (ordinary convergence in $\mathbb{R}$).

Proof: Let , $v \in A_\epsilon(T)$. Then $\| u - Tu \|^* \leq \epsilon$ and $\| v - Tv \|^* \leq \epsilon$. Using the multiplicative triangle inequality, $\| u - v \|^* \leq \| u - Tu \|^* \cdot \| Tu - Tv \|^* \cdot \| Tv - v \|^* \leq \epsilon^2 \cdot \| Tu - Tv \|*$.

Now treat the three cases separately.

If T is Banach type, then $\| u - v \|^* \leq \epsilon^2 \cdot (\| u - v \|^*)^\alpha \Longrightarrow (\| u - v \|^*)^{1-\alpha} \leq t^2 \Longrightarrow \| u - v \|^* \leq \epsilon^{\frac{2}{1-\alpha}}$.

If T is Kannan type, then $\| Tu - Tv \|^* \leq (\epsilon \cdot \epsilon)^\alpha = \epsilon^{2\alpha}$, hence $\| u - v \|^* \leq \epsilon^2 \cdot \epsilon^{2\alpha} = \epsilon^{2+2\alpha}$.

If T is Chatterjea type , then $\| u - Tv \|^* \leq \| u - v \|^* \cdot \| v - Tv \|^* \leq \epsilon . \| u - v \|^* \cdot$, and similarly $\| v - Tu \|^* \leq \| v - u \|^* \cdot \| u - Tu \|^* \leq \epsilon . \| u - v \|^*$. Hence $\| Tu - Tv \|^* \leq (\| u - v \|^* \epsilon \cdot \| u - v \|^* \epsilon)^\alpha = \epsilon^{2\alpha} \cdot (\| u - v \|^*)^{2\alpha}$. Substituting into the first estimate gives $\| u - v \|^* \leq \epsilon^2 \cdot \epsilon^{2\alpha} \cdot (\| u - v \|^*)^{2\alpha}$,hence $\| u - v \|^* \leq \epsilon^{\frac{2+2\alpha}{1-2\alpha}}$. Taking the supremum over all $u, v \in A_\epsilon(T)$ yields the stated bound for $\delta_m(A_\epsilon(T))$. Since each exponent is positive , $\delta_m(A_{\epsilon_n}(T)) \to 1$ whenever $\epsilon_n \to 1^+$.

Lemma 4.7 (Approximate fixed points always exist).

Let S be a multiplicative normed linear space and let $T: S \to S$ be any contraction type from Definition 4.1. Then for every $\epsilon > 1$, the set of ϵ-approximate fixed points $A_\epsilon(T)$ is nonempty.

Proof. Define $= \inf_{u \in S} \| u - Tu \|^*$. Clearly $m \geq 1$. We show $m = 1$. Then, by definition of infimum, for any $\epsilon > 1$, there must be some u with $\| u - Tu \|^* < \epsilon$, i.e., $u \in A_\epsilon(T)$.

Banach case: From the contraction inequality at (Tu, u), we get $\| T^2u - Tu \|^* \leq (\| u - Tu \|^*)^\alpha$.

$m \leq \| T^2u - Tu \|^* \leq (\| u - Tu \|^*)^\alpha, \forall u$. Then taking infimum over u, we have $m \leq m^\alpha$. If $m > 1$, then $m^\alpha < m$ (since $\alpha < 1$), a contradiction. Hence $m = 1$.

Kannan case: Applying the condition at (u, Tu), set $a = \| u - Tu \|^*$ and $= \| Tu - T^2u \|^*$. Then $b \leq (a \cdot b)^\alpha \Rightarrow b \leq a^{\frac{\alpha}{1-\alpha}}$. Let $\beta = (\frac{\alpha}{1-\alpha}) < 1$. Taking infima gives $\leq m^\beta$, and the same contradiction forces $m = 1$.

Chatterjea case: Same as Kannan, using $\| Tu - Tu \|^* = 1$ and the multiplicative triangle inequality $\| u - T^2u \| * \leq a \cdot b$, which yields the same inequality $b \leq (a \cdot b)^\alpha$.

Thus, in all cases $m = 1$, so $A_\epsilon(T) \neq \emptyset$ for every $\epsilon > 1$. No iteration or sequence construction is used.

Remark 4.8: This fills the non-emptiness gap in Theorem 5.1 without relying on Picard iterations, staying within the paper's non-iterative framework.

FIXED POINT THEOREMS

We now apply the multiplicative Cantor intersection theorem to obtain fixed point results for the mapping considered above.

Theorem 5.1 (Fixed Point Theorem for Multiplicative Contractions)

Let $(S, \|\cdot\|^*)$ be a multiplicative Banach space and let $T: S \to S$ a continuous mapping satisfying one of the contraction conditions in definition 4.1 . Assume that the approximate fixed-point sets $A_\epsilon(T) = \{u \in S: \| u - Tu \|^* \leq \epsilon\}$ are non-empty for every $\epsilon > 1$. Then T has a unique fixed point.

Proof: Choose a decreasing sequence $\{\epsilon_n\}$ satisfying $1 < \epsilon_{n+1} < \epsilon_n$ such that $\epsilon_n \to 1$ and for each n, define $A_n = A_{\epsilon_n}(T) = \{u \in S: \| u - Tu \|^* \leq \epsilon_n\}$. By lemma 4.5 and 4.7, each A_n is nonempty, multiplicative closed and $A_{n+1} \subseteq A_n$. By theorem 4.6, we have $\delta_m(A_n) \to 1$. Hence $\{A_n\}$ is a nested sequence of nonempty multiplicatively closed subsets whose multiplicative diameters converge to 1. Therefore, Multiplicative Cantor Intersection Theorem gives $\bigcap_{n=1}^{\infty} A_n = \{p\}$ for some unique point $p \in S$. Since $p \in A_n$ for every n, $\| p - Tp \|^* \leq \epsilon_n (n \geq 1)$, applying limit $\| p - Tp \|^* = 1$. By definition of multiplicative norm, $Tp = p$. Thus p be another fixed point of $\boldsymbol{T}$.

To prove uniqueness, suppose that q is another fixed point. Then $Tq = q$. Applying the corresponding contraction condition to the pair (p, q) gives:

Banach case: $\|p - q\|^* \leq (\|p - q\|^*)^\alpha$

Kannan case: $\|p - q\|^* \leq (\|p - Tp\|^*. \|q - Tq\|^*)^\alpha = 1$.

Chatterjea case: $\|p - q\|^* \leq (\|p - Tq\|^*. \|q - Tp\|^*)^\alpha = (\|p - q\|^*)^{2\alpha}$

Since $0 < \alpha < 1$, each inequality forces $\|p - q\|^* = 1$ and consequently $p = q$.

Therefore, T has a unique fixed point.

Remark 5.2. Unlike the classical Banach, Kannan, and Chatterjea fixed point theorems, the proof of Theorem 5.1 does not rely on the construction of an iterative sequence. Instead, the existence and uniqueness of the fixed point follow from the geometric properties of the approximate fixed-point sets, namely their multiplicative closedness, nesting, and the diameter estimates established in Section 4, together with the Multiplicative Cantor Intersection Theorem.

Example 5.3: Let $S = \mathbb{R}$ with $\| u \|^* = e^{|u|}$. Then $(S, \|\cdot\|^*)$ is a multiplicative Banach space.

(Banach type). With $T(u) = \frac{u}{2}$: $\| Tu - Tv \|^* = e^{\frac{|u-v|}{2}} = (\| u - v \|^*)^{\frac{1}{2}}$, so $\alpha = \frac{1}{2}$. Since $A_\epsilon(T) = \{u: | u | \leq 2 ln \epsilon\}$ shrinks to a point as $\epsilon \to 1^+$,the Cantor intersection theorem gives the unique fixed point $u = 0$.

(Kannan type). With $T(u) = \frac{u}{10}$: $\| Tu - Tv \|^* = e^{\frac{|u-v|}{10}} = (\| u - v \|^*)^{\frac{1}{10}}$ while $\| u - Tu \|^*. \| v - Tv \| * = e^{\frac{9}{10}(|u|+|v|)}$. Since $| u - v | \leq | u | + | v |$, exponentiating $\frac{1}{10} | u - v | \leq \frac{1}{9} \cdot \frac{9}{10}(| u | + | v |)$gives $\| Tu - Tv \|^* \leq (\| u - Tu \|^* \| v - Tv \|^*)^{\frac{1}{9}}$, so $\alpha = \frac{1}{9}$. Since $A_\epsilon(T) = \left\{u: \middle| u \middle| \leq \frac{10}{9} l n \epsilon \right\}$ shrinks to a point as $\epsilon \to 1^+$,the Cantor intersection theorem gives the unique fixed point $u = 0$.

(Chatterjea type). With $T(u) = \frac{u}{10}$: $u - v = \frac{1}{10}[(u - 10v) - (v - 10u)]$ and the triangle inequality, $\frac{1}{10} | u - v | \leq \frac{1}{11}(| u - \frac{v}{10} | + | v - \frac{u}{10} |)|)$, so $\| Tu - Tv \|^* \leq (\| u - Tv \|^* \| v - Tu \|^*)^{\frac{1}{11}}$, so $\alpha = \frac{1}{11}$. Since $A_\epsilon(T) = \left\{u: \middle| u \middle| \leq \frac{10}{9} l n \epsilon \right\}$ shrinks to a point as $\epsilon \to 1^+$,the Cantor intersection theorem gives the unique fixed point $u = 0$.

APPLICATION TO A VOLTERRA INTEGRAL EQUATION

The examples in Section 5 demonstrate our theory on the simplest possible space, $\mathbb{R}$ with the multiplicative norm $e^{\|\cdot\|^*}$. We now demonstrate that the same theory applies to an infinite-dimensional setting by studying a nonlinear Volterra integral equation. Unlike the classical approach based on successive approximations, our existence and uniqueness result follows directly from the Multiplicative Cantor Intersection Theorem and the diameter estimates established in Section 4.

Let $X = C([a, b], \mathbb{R})$ be the Banach space of all continuous real valued functions on $[a, b]$ with the usual supremum norm $\|f\|_\infty = Sup_{t \in [a,b]} |f(t)|$. Define a multiplicative norm on X by $\|f\|^* = e^{\|f\|_\infty}$.

Proposition 6.1. $(X, \|.\|^*)$ is a multiplicative Banach space, and $\|f\|^* = e^{\|f\|_\infty}$, for every $f \in X$.

Proof. Since $\|f\|_\infty \geq 0$, implies $\|f\|^* = e^{\|f\|_\infty} \geq 1$, with equality if and only if $f = 0$.

For any scalar α, $\|\alpha f\|^* = e^{\|\alpha f\|_\infty} = \left(e^{\|f\|_\infty}\right)^{|\alpha|}$.

Also $\|f + g\|^* = e^{\|f+g\|_\infty} \leq e^{\|f\|_\infty}. e^{\|g\|_\infty} = \|f\|^*. \|g\|^*$

Hence $\|.\|^*$ is a multiplicative norm. Since $(X, \|f\|_\infty)$ is complete, Proposition 2.1 gives that $(X, \|.\|^*)$ is multiplicatively complete.

Now consider the nonlinear Volterra integral equation

$$u(t) = g(t) + \lambda \int_a^t K(t, s, u(s))\, ds, t \in [a, b].$$

We assume:

(H1) g is continuous on $[a, b]$;

(H2) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous,

(H3) here exists $L > 0$ such that

$| K(t, s, u) - K(t, s, v) | \le L \mid u - v \mid$ for all t, s, u, v

(H4) $|\lambda|$ L(b − a) < 1..

Define the operator T on X by

$$(Tu)(t) = g(t) + \lambda \int_a^t K(t, s, u(s))ds, t \in [a, b].$$

Under (H1)-(H3), T maps X into itself.

Lemma 6.2. Under (H1)-(H4), T is a multiplicative Banach contraction with contraction constant $\alpha = |\lambda| L(b - a)$.

Proof. For any $u, v \in X$, $|(Tu)(t) - (Tv)(t)| \le \int_a^t |\lambda| \, |K(t, s, u(s)) - K(t, s, v(s))| ds$. $\le |\lambda| L \int_a^t |u(s) -, u(t))| ds \le |\lambda| L(b - a) \|u - v\|_\infty$

Taking Supremum, we have $\|Tu - Tv\|_\infty \le |\lambda| L(b - a) \|u - v\|_\infty$

Exponentiating yields, $\|Tu - Tv\|^* \le (\|u - v\|^*)^{|\lambda| L(b-a)}$. Since $|\lambda|$L(b − a) < 1 , T is a multiplicative Banach contraction.

Theorem 6.3 (Existence and Uniqueness). Under (H1)-(H4), the nonlinear Volterra equation has a unique continuous solution.

Proof. By Proposition 6.1, X is a multiplicative Banach space. By Lemma 6.2, T is a multiplicative Banach contraction. Therefore, all hypotheses of Theorem 5.1 are satisfied. Hence the Multiplicative Cantor Intersection Theorem guarantees that the nested approximate fixed-point sets $A_\epsilon(T) = \{u \in X: \|u - Tu\|^* \le \epsilon\}$ have singleton intersection. Consequently, T possesses a unique fixed point $u^* \in X$, which is precisely the unique solution of the Volterra integral equation.

Corollary 6.4 (Error Bound). Let u^* denote the unique solution and suppose that $u \in X$ satisfies $\|u - Tu\|^* \le r, r > 1$. Then $\|v - u^*\|^* \le r^{\frac{1}{1-\alpha}}$, where α = $|\lambda|$L(b − a).

Proof. Since $Tu^* = u^*$, the contraction condition gives

$\|u - u^*\|^* \le \|u - Tu\|^* \|Tu - Tu^*\|^* \le r \, . (\|u - u^*\|)^\alpha$. So $(\|u - u^*\|)^{1-\alpha} \le r$

Giving $(\|v - u^*\|^*) \le r^{\frac{1}{1-\alpha}}$. Hence the multiplicative distance between the approximate solution u and the exact solution u* is bounded by $r^{\frac{1}{1-\alpha}}$.

CONCLUSION

In this paper, we developed a non-iterative set-theoretic approach to fixed point theory in multiplicative Banach spaces. By establishing multiplicative Cantor and Baire category theorems, we derived explicit diameter estimates for approximate fixed-point sets corresponding to Banach, Kannan, and Chatterjea type contractions. These estimates, together with the Multiplicative Cantor Intersection Theorem, yielded existence and uniqueness results without using iterative sequences. Finally, the applicability of the theory was demonstrated through a nonlinear Volterra integral equation, illustrating the effectiveness of the proposed framework in infinite-dimensional multiplicative Banach spaces.

ACKNOWLEDGEMENT:

The authors would like to thank everyone who contributed, directly or indirectly, to the completion of this research. Their support and encouragement are greatly appreciated.

***Source of support*:** None

***Conflicts of Interest*:** None.

REFERENCES

Badshah-e-Rome, V., & Sarwar, M. (2017). Extensions of the Banach contraction principle in multiplicative metric spaces. *Vojnotehnicki Glasnik, 65*, 346–358. DOI: https://doi.org/10.5937/vojtehg65-13342

Boyd, D. W., & Wong, J. S. W. (1968). Another proof of the contractive mapping principle. *Canadian Mathematical Bulletin, 11*, 605–606. DOI: https://doi.org/10.4153/CMB-1968-075-1

Chatterjea, S. K. (1972). Fixed point theorems. *C. R. Acad. Bulgare Sci., 25*, 727–730.

Chugh, R., Nandal, A., & Kumar, N. (2017). Multiplicative normed linear space and its topological properties. *Indian Journal of Science and Technology, 10*, 1–9. DOI: https://doi.org/10.17485/ijst/2017/v10i10/74294

Kannan, R. (1969). On certain sets and fixed point theorems. *Revue Roumaine de Mathématiques Pures et Appliquées, 14*, 51–54.

Lahiri, B. K., Das, P., & Dey, L. K. (2011). Cantor's theorem in 2-metric spaces and its applications to fixed point problems. *Taiwanese Journal of Mathematics, 15*, 337–352. DOI: https://doi.org/10.11650/twjm/1500406178

Özavşar, M., & Çevikel, A. C. (2017). Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *Journal of Engineering Technology and Applied Sciences, 2*(2), 65–79. DOI: https://doi.org/10.30931/jetas.338608