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On S—3 Like Five-Dimensional Finsler Spaces

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ABSTRACT

In 1977, M. Matsumoto and R. Miron [1] constructed an orthonormal frame for an n−dimensional Finsler space called 'Miron frame'. M. Matsumoto [2,3] proved that in a three-dimensional Berwald space, all the main scalars are h−covariant constants and the h−connection vector vanishes. He also proved that in a three-dimensional Finsler space satisfying T−condition, all the main scalars are function of position only and the v−connection vector vanishes [2, 4]. M. K. Gupta and P. N. Pandey [5] proved that in an S− like four-dimensional Berwald space satisfying T−condition, all the main scalars are constants and the h− and v−connection vector vanish. The purpose of the present paper is to generalize these results for an S − 3 like five-dimensional Finsler space.

1. Preliminaries

Let M^5 be a five-dimensional smooth manifold and $F^5 = (M^5, L)$ be a five dimensional Finsler space equipped with a metric function $L(x, y)$ on M^5. The normalized supporting element, the metric tensor, the angular metric tensor and

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Cartan tensor are defined by $l_i = \dot{\partial}_i L, g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ and $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ respectively. The Cartan connection in the Finsler space is given as $C\Gamma = \left(F^i_{jk}, G^i_j, C^i_{jk}\right)$. The h − and v − covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$X_{i|j} = \partial_j X_i - \left(\dot{\partial}_h X_i\right) G^h_j - F^r_{ij} X_r \tag{1.1}$$

and

$$X_i|_j = \dot{\partial}_j X_i - C^r_{ij} X_r. \tag{1.2}$$

The Miron frame for a five-dimensional Finsler space is constructed by the unit vectors $\left(e^i_{1)}, e^i_{2)}, e^i_{3)}, e^i_{4)}, e^i_{5)}\right)$. The first vector $e^i_{1)}$ is the normalized torsion vector $m_i = C^i/C$, where C is the length of the torsion vector C^i. The third $e^i_{3)} = n^i$, the fourth $e^i_{4)} = p^i$ and the fifth are constructed by the method of Matsumoto and Miron [6].

With respect to this frame, the scalar components of an arbitrary tensor T^i_j are defined by

$$T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\beta)}. \tag{1.3}$$

From this, we get

$$T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j}. \tag{1.4}$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$. Therefore, we get

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j + q_i q_j. \tag{1.5}$$

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}/L$ be the scalar components of the h − and v − covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ respectively of the vector $e^i_{\alpha)}$, then,

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}, \tag{1.6}$$

and

$$L e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)_j}. \tag{1.7}$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called $h-$ and $v-$ connection scalars respectively and are positively homogeneous of degree '0' in y.

Orthogonality of the Miron frame yields

$$H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma} \text{ and } V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}.$$

Also, we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}$ [2].

Now, we define Finsler vector fields:

$$h_i = H_{2)3\gamma}e_{\gamma)i},\quad J_i = H_{2)4\gamma)}e_{\gamma)i},\quad k_i = H_{2)5\gamma)}e_{\gamma)i},$$
$$h'_i = H_{3)4\gamma}e_{\gamma)i},\quad J'_i = H_{3)5\gamma)}e_{\gamma)i},\quad k'_i = H_{4)5\gamma)}e_{\gamma)i},$$

and

$$u_i = V_{2)3\gamma}e_{\gamma)i},\quad v_i = V_{2)4\gamma)}e_{\gamma)i},\quad w_i = V_{2)5\gamma)i},$$
$$u'_i = V_{3)4\gamma}e_{\gamma)_i},\quad v'_i = V_{3)5\gamma)}e_{\gamma)i},\quad w'_i = V_{4)5\gamma)}e_{\gamma)i}.$$

The vector fields h_i, J_i, k_i, J'_i, and k'_i are called $h-$connection vectors and the vector fields u_i, v_i, w_i, u'_i, v'_i and w'_i are called v–connection vectors. The scalars $H_{2)3\gamma}$, $H_{2)4\gamma}$, $H_{2)5\gamma}$, $H_{3)4\gamma}$, $H_{3)5\gamma}$, $H_{4)5\gamma}$ and $V_{2)3\gamma}$, $V_{2)4\gamma}$, $V_{2)5\gamma}$, $V_{3)4\gamma}$, $V_{3)5\gamma}$, $V_{4)5\gamma}$ are considered as the scalar components h_γ, J_γ, K_γ, h'_γ, J'_γ, k'_γ, and u_γ, v_γ, w_γ, u'_γ, v'_γ, w'_γ of the $h-$ and $v-$ connection vectors respectively with respect to the orthonormal frame.

Let $C_{\alpha\beta\gamma}$ are the scalar components of LC_{ijk}, then

$$LC_{ijk} = C_{\alpha\beta\gamma}\, e_{\alpha)i}\, e_{\beta)i}\, e_{\gamma)k}. \tag{1.8}$$

The main scalars of a five-dimensional Finsler space are given by [7]

$C_{222} = H$, $C_{233} = I$, $C_{244} = K$, $C_{333} = J$, $C_{344} = J'$,
$C_{444} = H'$, $C_{334} = I'$, $C_{234} = K'$, $C_{255} = M$, $C_{355} = J''$,
$C_{455} = M'$, $C_{555} = H''$, $C_{335} = I''$, $C_{445} = K''$, $C_{235} = N$,
$C_{245} = N'$, $C_{345} = M''$.

We also have

$$C_{223} = -(J + J' + J''),\ C_{224} = -(H' + I' + M'),$$
$$C_{225} = -(H'' + I'' + K''),\ H + I + K + M = LC. \tag{1.9}$$

LC is called the unified main scalar.

Taking $h-$ covariant differentiation of (1.4), we get

$$T^i_{j|k} = \left(\delta_k T_{\alpha\beta}e^i_{\alpha)}e_{\beta)j} + T_{\alpha\beta}e^i_{\alpha)|k}e_{\beta)j} + T_{\alpha\beta}e^i_{\alpha)}e_{\beta)j|k}, \tag{1.10}$$

where $\delta_k = \partial_k G^r_k \dot{\partial}_r$. If $T_{\alpha\beta,\gamma}$ are scalar components of $T^i_{j|k}$, i.e.

$$T^i_{j|k} = T_{\alpha\beta,\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}, \tag{1.11}$$

then we obtain

$$T_{\alpha\beta,\gamma} = \left(\delta_k . T_{\alpha\beta}\right)e^k_{\gamma)} + T_{\mu\beta}H_{\mu)\alpha\gamma} + T_{\alpha\mu}H_{\mu)\beta\gamma}. \tag{1.12}$$

Similarly, if $T_{\alpha\beta;\gamma}$ are scalar components of $LT^i_j|_k$, i.e.

$$LT^i_{j|k} = T_{\alpha\beta;\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}, \tag{1.13}$$

then, we get

$$T_{\alpha\beta;\gamma} = L\left(\dot{\partial}_k T_{\alpha\beta}\right)e^k_{\gamma)} + T_{\mu\beta}V_{\mu)\alpha\gamma} + T_{\alpha\mu}V_{\mu)\beta\gamma.} \tag{1.14}$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are respectively called $h-$ and $v-$scalar derivatives of scalar components $T_{\alpha\beta}$ of T.

2. T–CONDITION

The tensor T_{hijk} defined by

$$T_{hijk} = LC_{hij|k} + C_{hij}l_k + C_{hik}l_j + C_{hkj}\, l_i + C_{kij}\, l_h, \tag{2.1}$$

is called T–tensor in a Finsler space. It is completely symmetric in its indices. A Finsler space is said to satisfy $T-$condition if the $T-$tensor T_{hijk} vanishes identically

We are conserned with the tensor $C_{hij|k}$. From (1.8) and (1.13), it follows that

$$L^2C_{hij|k} + LC_{hij}l_k = C_{\alpha\beta\gamma;\delta}e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k},$$

which implies

$$L^2C_{hij}|_k = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{|\delta})e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k}, \tag{2.2}$$

Therefore the scalar components $T_{\alpha\beta\gamma\delta}$ of LT_{hijk} are given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} + \delta_{|\alpha}C_{\beta\gamma\delta} + \delta_{|\beta}C_{\alpha\gamma\delta} + \delta_{|\gamma}C_{\alpha\beta\delta}.$$

From $T_{hijk}.l^k = 0$, we have $T_{\alpha\beta\gamma|\delta} = 0$. Thus, the surviving components $T_{\alpha\beta\gamma\delta}$ are only

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta};\quad \alpha,\ \beta, \gamma, \delta = 2,\ 3,\ 4,\ 5. \tag{2.3}$$

Using (1.14), the explicit forms of $C_{\alpha\beta\gamma;\delta}$ are obtained as follows:

a) $C_{222;\delta} = H_{;\delta} + 3(J + J' + J'')u_\delta + 3(H' + I' + M')v_\delta + 3(H'' + I'' + K'')w_\delta$,

b) $C_{223;\delta} = -(J + J' + J'')_{;\delta} + (H - 2I)u_\delta - 2K'v_\delta - 2Nw_\delta + (H' + I' + M')u'_\delta + (H'' + I'' + M'')v'_\delta$,

c) $C_{224;\delta} = -(H' + I' + M')_{;\delta} - 2K'u_\delta + (H - 2K)v_\delta - 2N'w_\delta - (J + J' + J'')u'_\delta + (H'' + I'' + M'')w'_\delta$,

d) $C_{225;\delta} = -(H'' + I'' + K'')_{;\delta} - 2Nu_\delta - 2N'v_\delta + (H - 2M)w_\delta - (J + J' + J'')v'_\delta - (H' + I' + M')w'_\delta$,

e) $C_{233;\delta} = I_{;\delta} - (3J + 2J' + 2J'')u_\delta - I'v_\delta - I''w_\delta - 2Nv'_\delta - 2K'u'_\delta$,

f) $C_{234;\delta} = K'_{;\delta} - (2I' + H' + M')u_\delta - (2J' + J + J'')v_\delta - M''w_\delta - (K - I)u'_\delta - N'v'_\delta - Nw'_\delta$,

g) $C_{235;\delta} = N_{;\delta} - (2I'' + H'' + K'')u_\delta - M''v_\delta - (J + J' + 2J'')w_\delta - N'u'_\delta - (M - I)v'_\delta + K'w'_\delta$,

h) $C_{244;\delta} = K_{;\delta} - J'u_\delta - (3H' + 2I' + 2M')v_\delta + 2K u'_\delta - K''w_\delta - 2N' w'_\delta$,

i) $C_{245;\delta} = N'_{;\delta} - M''u_\delta - (H'' + I'' + 2K'')v_\delta + N u'_\delta - (H' + I' + 2M')w_\delta + K' v'_\delta + (K - M) w'_\delta$,

j) $C_{255;\delta} = M_{;\delta} - J''u_\delta - M'v_\delta - (3H'' + 2I'' + 2K'')w_\delta + 2N v'_\delta + 2N' w'_\delta$, (2.4)

k) $C_{333;\delta} = J_{;\delta} + 3(Iu_\delta - I'u'_\delta - I''v'_\delta)$,

l) $C_{334;\delta} = I'_{;\delta} + 2K'u_\delta + Iv_\delta + (J - 2J')u'_\delta - 2M'' v'_\delta - I'' w'_\delta$

m) $C_{335;\delta} = I''_{;\delta} + 2Nu_\delta - 2M''u'_\delta + (J - 2J'')v'_\delta + Iw_\delta + I'w'_\delta$,

n) $C_{344;\delta} = J'_{;\delta} + Ku_\delta + 2K'v_\delta - (H - 2I')u'_\delta - K''v'_\delta - 2M''w'_\delta$,

o) $C_{345;\delta} = M''_{;\delta} + N'u_\delta + Nv_\delta + (I'' - K'')u'_\delta + K'w_\delta + (I' - M')v'_\delta + (J' - J'')w'_\delta$,

p) $C_{355;\delta} = J''_{;\delta} + Mu_\delta - M'u'_\delta + 2Nw_\delta - (H'' - 2I'') v'_\delta + 2M''w'_\delta$,

q) $C_{444;\delta} = H'_{;\delta} + 3(Kv_\delta + J'u'_\delta - K''w'_\delta)$,

r) $C_{445;\delta} = K''_{;\delta} + 2N'v_\delta + 2M''u'_\delta + Kw_\delta + J'v'_\delta + (H' - 2M')w'_\delta$,

s) $C_{455;\delta} = M'_{;\delta} + Mv_\delta + J''u'_\delta + 2N'w_\delta + 2M'v'_\delta - (H'' - 2K'')w'_\delta$,

t) $C_{555;\delta} = \;'' + 3(Mw_\delta + J''v'_\delta + M'w'_\delta)$,

where $H_{;\delta}$, for instance is the v−scalar derivative of the single scalar H, namely $H_{;\delta} = L\left(\dot{\partial}_i H\right)e^i_{\delta)}$. From (1.9) and (2.4), we get

$C_{222;\delta} + C_{233;\delta} + C_{244;\delta} + C_{255;\delta} = (H + I + K + M)_{;\delta} = (LC)_{;\delta}$,

$C_{322;\delta} + C_{333;\delta} + C_{344;\delta} + C_{355;\delta} = (H + I + K + M) u_\delta = (LC)u_\delta$,

$C_{422;\delta} + C_{433;\delta} + C_{444;\delta} + C_{455;\delta} = (H + I + K + M) v_\delta = (LC)v_\delta$, (2.5)

$C_{522;\delta} + C_{533;\delta} + C_{544;\delta} + C_{555;\delta} = (H + I + K + M) w_\delta = (LC)w_\delta$.

Thus, from (2.3), (2.4) and (2.5), we have

Theorem 2.1. In a five-dimensional Finsler space satisfying T−condition, the v−connection vectors u_i, v_i and wi vanish identically. Also main scalar H and the unified main scalar LC are v−covariant constants (functions of position only). Furthermore, if v−connection vectors u'_i, v'_i and w'_i vanishes. Then all the main scalars are functions of position only.

3. Berwald Space

A Berwald space is characterized by $C_{hij|k} = 0$. From (1.8) and (1.11), it follows that

$$LC_{hij|k} = C_{\alpha\beta\gamma,\delta}\, e_{\alpha)h}\, e_{\beta)i}\, e_{\gamma)j}\, e_{\delta)k},$$

where $C_{\alpha\beta\gamma,\delta}$ are given by

$$C_{\alpha\beta\gamma,\delta} = (\delta_k . C_{\alpha\beta\gamma})\, e^k_{\delta)} + C_{\mu\beta\gamma}\, H_{\mu\alpha\delta} + C_{\alpha\mu\gamma}H_{\mu})_{\beta\delta} + C_{\alpha\beta\mu}H_{\mu})_{\gamma\delta}.$$

The explicit forms of $C_{\alpha\beta\gamma,\delta}$ are obtained as follows:

a) $C_{1\beta\nu,\delta} = 0$,

b) $C_{222,\delta} = H_{,\delta} + 3(J + J' + J'')h_\delta + 3(H' + I' + M')J_\delta + 3(H'' + I'' + K'')k_\delta$,

c) $C_{223,\delta} = -(J + J' + J'')_{,\delta} + (H - 2I)h_\delta - 2K'J_\delta - 2Nk_\delta + (H' + I' + M')h'_\delta + (H'' + I'' + M'') j'_\delta$,

d) $C_{224,\delta} = -(H' + I' + M')_{,\delta} - 2K'h_\delta + (H - 2K)J_\delta - 2N'k_\delta - (J + J' + J'')h'_\delta + (H'' + I'' + K'') k'_\delta$,

e) $C_{225,\delta} = -(H'' + I'' + K'')_{,\delta} - 2Nh_\delta - 2N'J_\delta + (H - 2M)k_\delta - (J + J' + J'')J'_\delta - (H' + I' + M')k'_\delta$,

f) $C_{233,\delta} = I_{,\delta} - (3J + 2J' + 2J'')h_\delta - I'J_\delta - I''k_\delta - 2N J'_\delta - 2K'h'_\delta$

g) $C_{234,\delta} = K'_{,\delta} - (2I' + H' + M')h_\delta - (2J' + J + J'')J_\delta - M''k_\delta - (K - I)h'_\delta - N'J'_\delta - Nk'_\delta$,

h) $C_{235,\delta} = N_{,\delta} - (2I'' + H'' + K'')h_\delta - M''J_\delta - (J + J' + 2J'')k_\delta - N'h'_\delta - (M - I)J'_\delta + K'k'_\delta$,

i) $C_{244,\delta} = K_{,\delta} - J'h_\delta - (3H' + 2I' + 2M')J_\delta + 2Kh'_\delta - K''k_\delta - 2N'k'_\delta$,

j) $C_{245,\delta} = N'_{,\delta} - M''h_\delta - (H'' + I'' + 2K'')J_\delta + Nh'_\delta - (H' + I' + 2M')k_\delta + K'J'_\delta + (K - M)k'_\delta$,

k) $C_{255,\delta} = M_{,\delta} - J''h_\delta - M'J_\delta - (3H'' + 2I'' + 2K'')k_\delta + 2NJ'_\delta + 2N'k'_\delta$,

l) $C_{333,\delta} = J_{,\delta} + 3(Ih_\delta - I'h'_\delta - I''J'_\delta)$,

m) $C_{334,\delta} = I'_\delta + 2K'h_\delta + IJ_\delta + (J - 2J')h'_\delta - 2M''J'_\delta - I''k'_\delta$, (3.2)

n) $C_{335,\delta} = I''_{,\delta} + 2Nh_\delta - 2M''h'_\delta + (J - 2J'')J'_\delta + Ik_\delta + I'k'_\delta$,

o) $C_{344,\delta} = J'_{,\delta} + Kh_\delta + 2K'J_\delta - (H - 2I')h'_\delta - K''J'_\delta - 2M''k'_\delta$,

p) $C_{345,\delta} = M''_{,\delta} + N'h_\delta + NJ_\delta + (I'' - K'')h'_\delta + K'k_\delta + (I' - M')J'_\delta(J' - J'')k'_\delta$,

q) $C_{355,\delta} = J''_{,\delta} + Mh_\delta - M'h'_\delta + 2Nk_\delta - (H'' - 2I'')J'_\delta + 2M''k'_\delta$,

r) $C_{444,\delta} = H'_{,\delta} + 3(KJ_\delta + J'h'_\delta - K''k'_\delta)$,

s) $C_{445,\delta} = K''_{,\delta} + 2N'J_\delta + 2M''h'_\delta + Kk_\delta + J'J'_\delta + (H' - 2M')k'_\delta$,

t) $C_{455,\delta} = M_{,\delta} + MJ_\delta + J''h'_\delta + 2N'k_\delta + 2M'J'_\delta - (H'' - 2K'')k'_\delta$,

u) $C_{555,\delta} = H''_{,\delta} + 3(Mk_\delta + J''J'_\delta + M'k'_\delta)$.

From (1.9) and (3.2), we get

$$
\begin{aligned}
&C_{322,\delta} + C_{333,\delta} + C_{344,\delta} + C_{355,\delta} = (H + I + K + M)u_\delta = (LC)h_\delta,\\
&C_{422,\delta} + C_{433,\delta} + C_{444,\delta} + C_{455,\delta} = (H + I + K + M)J_\delta = (LC)J_\delta,\\
&C_{522,\delta} + C_{533,\delta} + C_{544,\delta} + C_{555,\delta} = (H + I + K + M)w_\delta = (LC)k_\delta,\\
&C_{222,\delta} + C_{233,\delta} + C_{244,\delta} + C_{255,\delta} = (H + I + K + M)_{,\delta} = (LC)_{,\delta}.
\end{aligned} \tag{3.3}
$$

Thus from (3.2) and (3.3), we get

Theorem 3.1. In a five-dimensional Berwald space, the h−connection vectors h_i, J_i and k_i vanish identically. Also main scalar H and the unified main scalar LC are h−covariant constants. Furthermore, if h−connection vector h'_i, J'_i and k'_i vanishes then all the main scalars are h−covariant constants.

4. v-Curvature Tensor

The v−curvature tensor is defined by

$$S_{hijk} = C^r_{hk}C_{ijr} - C^r_{hj}C_{ikr}. \tag{4.1}$$

The scalar components $S_{\alpha\beta\gamma\delta}$ of L^2S_{hijk} are given by

$$L^2S_{hijk} = S_{\alpha\beta\gamma\delta}\, e_{\alpha)h}\, e_{\beta)i}\, e_{\gamma)j}\, e_{\delta)k}. \tag{4.2}$$

Since S_{hijk} is skew-symmetric in h and i as well as j and k and $S_{0ijk} = S_{hi0k} = 0$, the surviving independent components of $S_{\alpha\beta\gamma\delta}$ are twenty, which are given by

$S_{2323} = C_{23\mu}C_{\mu32} - C_{22\mu}C_{\mu33}$
$= I^2 + I'^2 + I''^2 + 2J^2 + +J'^2 + J''^2 + K'^2 + N'^2 + 3JJ' + 3JJ'' + 2J'J'' + H'I' + M'I' + H''I'' + K''I'' - HI$,

$S_{2324} = C_{23\mu}C_{\mu32} - C_{22\mu}C_{\mu34}$
$= J^2 + J'^2 + J''^2 + I^2 + +K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JI' + J'I' + J''I' + H'I' + I'J' + M'J' + H''M'' + I''M'' + K''M''$,

$S_{2325} = C_{23\mu}C_{\mu32} - C_{22\mu}C_{\mu35}$
$= J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + I''J + I''J' + I''J'' + H'M'' + I'M'' + M'M'' + H''J'' + I''J'' + J''K'' - HN'$,

$S_{2334} = C_{23\mu}C_{\mu32} - C_{23\mu}C_{\mu34}$
$= J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + J'K + J'K' + J''K' - II' - J'K'' - N'M''$,

$S_{2335} = C_{23\mu}C_{\mu32} - C_{23\mu}C_{\mu35}$
$= J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JN' + J'N' - II'' - K'M''$,

$S_{2345} = C_{23\mu}C_{\mu32} - C_{24\mu}C_{\mu35}$
$= J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + H'N' + I'N' + M'N' - K'I'' - KM - N'J''$,

$S_{2424} = C_{\mu24}C_{\mu42} - C_{\mu22}C_{\mu44}$
$= J^2 + J'^2 + J''^2 + K^2 + K'^2 + K''^2 + N'^2 + H'^2 + 2JJ'' + 3J'J'' + 3JJ'' + H'I' + M'H' + H''K'' + I''K'' - HK$,

$S_{2434} = C_{\mu24}C_{\mu42} - C_{\mu23}C_{\mu44}$
$= H'^2 + I'^2 + M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + JK + J'K + J''K - IJ' - H'K' - N'K''$,

$S_{2435} = C_{\mu24}C_{\mu42} - C_{\mu23}C_{\mu45}$
$= H'^2 + I'^2 + 2H'I' + M'^2 + 2H'M' + 2I'M' + K'^2 + K^2 + N'^2 + JN' + J'N' + J''N' - IM'' - K'K'' - N'M'$,

$S_{2445} = C_{\mu24}C_{\mu42} - C_{\mu24}C_{\mu45}$
$= H'^2 + I'^2 + M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + H'N' + I'N' + M'N' - K'M'' - KK'' - N'M'$,

$S_{2525} = C_{\mu25}C_{\mu52} - C_{\mu22}C_{\mu55}$
$= H''^2 + I''^2 + K''^2 + J''^2 + H'^2 + N'^2 + M'^2 + N^2 + M^2 + 2H''I'' + 2H''K'' + 2I''K'' + JJ'' + J'J'' + H'M' + I'M' + I''H'' + K''H'' - HM$,

$S_{2535} = H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JM + J'M + J''M + IJ'' - K'M' - N'H''$,

$S_{2545} = H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' - H'M - I'M - MM' - K'J'' - KM' - N'H''$,

$S_{2534} = H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JN' + J'N' + J''N' - IM'' - K'K'' - NM'$,

$S_{3434} = K'^2 + I^2 + J^2 + M''^2 - IK - JJ' - H'I' - I''K''$,

$S_{3445} = K'^2 + I^2 + J^2 + M''^2 - K'N' - I'M'' - J'K'' - M'M''$,

$S_{3535} = N^2 + I''^2 + M''^2 + J''^2 - IM - JJ' - I'M' - I''H''$,

$S_{3534} = N^2 + I''^2 + M''^2 + J''^2 - IN' - JM'' - I'K'' - I''M'$,

$S_{3545} = N^2 + I''^2 + M''^2 + J''^2 - K'M - I'J'' - J'M' - M''H''$,

$S_{4545} = N'^2 + M''^2 + K''^2 + M'^2 - KM - J'J'' - H'M' - K''H''$.

A Finsler space $F^n (n \geq 4)$ is called $S - 3$ like, if there exists a scalar S such that the curvature tensor S_{hijk} of F^n is written in the form

$$L^2S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij})$$

$= S[(m_h m_j + n_h n_j + p_h p_j + q_h q_j)(m_i m_k + n_i n_k + p_i p_k + q_i q_k) - (m_h m_k + n_h n_k + p_h p_k + q_h q_k)(m_i m_j + n_i n_j + p_i p_j + q_i q_j)]$

$= S[(m_h n_i - m_i n_h)(m_j n_k - m_k n_j) + (m_h p_i - m_i p_h)(m_j p_k - m_k p_j) + (m_h q_i - m_i q_h)(m_j q_k - m_k q_j) + (n_k p_j - n_j p_k)(n_i p_h - nhpi) + (n_i q_h - n_h q_i)(n_k q_j - n_j q_k) + (q_i p_h - p_i q_h)(q_k p_j - p_k q_j)]$. (4.3)

This implies that the scalar components are

$S_{2323} = S$, $S_{2324} = 0$, $S_{2325} = 0$, $S_{2334} = 0$, $S_{2335} = 0$,
$S_{2345} = 0$, $S_{2424} = S$, $S_{2434} = 0$, $S_{2435} = 0$, $S_{2445} = 0$,
$S_{2525} = S$, $S_{2535} = 0$, $S_{2545} = 0$, $S_{2534} = 0$, $S_{3434} = S$,
$S_{3445} = 0$, $S_{3535} = S$, $S_{3534} = 0$, $S_{3545} = 0$, $S_{4545} = S$.

M. Matsumoto [8] proved that the v−curvature S of an $S - 3$ like Finsler space is a function of position only. Therefore in $S - 3$ like five-dimensional Finsler space, twenty functions

$I^2 + I'^2 + I''^2 + 2J^2 + +J'^2 + J''^2 + K'^2 + N'^2 + 3JJ' + 3JJ'' + 2J'J'' + H'I'+M'I' + H''I'' + K''I'' - HI,$

$J^2 + J'^2 + J''^2 + I^2 + +K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JI' + J'I' + J''I'+H'I' + I'J' +M'J' + H''M'' + I''M'' + K''M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + I''J + I''J' + I''J''+H'M'' + I'M'' +M'M'' + H''J'' + I''J'' + J''K'' - HN',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + J'K + J'K' + J''K'-II' - J'K'' - N'M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JN' + J'N'-II'' - K'M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + H'N' + I'N' + M'N' - K'I'' - KM - N'J'',$

$J^2 + J'^2 + J''^2 + K^2 + K'^2 + K''^2 + N'^2 + H'^2 + 2JJ'' + 3J'J'' + 3JJ'' + H'I' +M'H' + H''K'' + I''K'' - HK,$

$H'^2 + I'^2 +M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + JK + J'K +J''K - IJ' - H'K' - N'K'',$

$H'^2 + I'^2 + 2H'I' +M'^2 + 2H'M' + 2I'M' + K'^2 + K^2 + N'^2 + JN' + J'N' +J''N' - IM'' - K'K'' - N'M',$

$H'^2 + I'^2 +M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + H'N' + I'N' +M'N' - K'M'' - KK'' - N'M',$

$H''^2 + I''^2 + K''^2 + J''^2 + H'^2 + N'^2 +M'^2 + N^2 +M^2 + 2H''I'' + 2H''K'' + 2I''K'' + JJ'' + J'J'' + H'M' + I'M' + I''H'' + K''H'' - HM,$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 +M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JM +J'M + J''M + IJ'' - K'M' - N'H'',$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 +M^2 + 2H''I'' + 2I''K'' + 2H''K'' - H'M -I'M -MM' - K'J'' - KM' - N'H'',$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 +M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JN' +J'N' + J''N' - IM'' - K'K'' - NM',$

$K'^2 + I'^2 + J'^2 +M''^2 - IK - JJ' - H'I' - I''K'',$

$K'^2 + I'^2 + J'^2 +M''^2 - K'N' - I'M'' - J'K'' -M'M'',$

$N^2 + I''^2 +M''^2 + J''^2 - IM - JJ' - I'M' - I''H'',$

$N^2 + I''^2 +M''^2 + J''^2 - IN' - JM'' - I'K'' - I''M',$

$N^2 + I''^2 +M''^2 + J''^2 - K'M - I'J'' - J'M' -M''H'',$

$N'^2 +M''^2 + K''^2 +M'^2 - KM - J'J'' - H'M' - K''H''.$

are functions of position only. In view of theorem (2.1) and equation (1.9), function H and $H + I + K + M$ are functions of position only in a five-dimensional Finsler space satisfying T−condition. Thus, in a $S - 3$ like Finsler space satisfying T−condition twenty two functions H, $H + I + K + M$ and aforesaid twenty functions are functions of position only. These twenty two functions are clearly independent and therefore the main scalars $H, H', H'', I, I', I'', J, J', J'', M, M', M'', K, K', K'', N$ and N' are functions of position only. Thus, we have

Theorem 4.1. In an S−3 like five-dimensional Finsler space satisfying T−condition, all the main scalars are functions of position only.

It is clear from (2.4) that if all the main scalars are functions of position only in a Finsler space satisfying T−condition, then the v−connection vectors u_i, v_i, w_i, u'_i, v'_i and w'_i vanish. This lead to:

Theorem 4.2. In an $S - 3$like five-dimensional Finsler space satisfying T−condition, the v−connection vectors ui, v_i, w_i, u'_i, v'_i and w'_i vanish identically.

A landsberg space is characterized by $C_{hij|k} = C_{hik|j}$. H. Yasuda [9] proved that in an $S - 3$like Landsberg space, the v−curvature S is constant. In view of this result, in an S−like five-dimensional Landsberg space, twenty independent functions

$I^2 + I'^2 + I''^2 + 2J^2 + +J'^2 + J''^2 + K'^2 + N'^2 + 3JJ' + 3JJ'' + 2J'J'' + H'I' +M'I' + H''I'' + K''I'' - HI,$

$J^2 + J'^2 + J''^2 + I^2 + +K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JI' + J'I' + J''I' +H'I' + I'J' +M'J' + H''M'' + I''M'' + K''M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + I''J + I''J' + I''J'' +H'M'' + I'M'' +M'M'' + H''J'' + I''J'' + J''K'' - HN',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + J'K + J'K' + J''K' -II' - J'K'' - N'M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + JN' + J'N' -II'' - K'M'',$

$J^2 + J'^2 + J''^2 + I^2 + K'^2 + N'^2 + 2JJ' + 2JJ'' + 2J'J'' + H'N' + I'N' +M'N' - K'I'' - KM - N'J'',$

$J^2 + J'^2 + J''^2 + K^2 + K'^2 + K''^2 + N'^2 + H'^2 + 2JJ'' + 3J'J'' + 3JJ'' + H'I' + M'H' + H''K'' + I''K'' - HK,$

$H'^2 + I'^2 + M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + JK + J'K + J''K - IJ' - H'K' - N'K'',$

$H'^2 + I'^2 + 2H'I' + M'^2 + 2H'M' + 2I'M' + K'^2 + K^2 + N'^2 + JN' + J'N' + J''N' - IM'' - K'K'' - N'M',$

$H'^2 + I'^2 + M'^2 + K'^2 + K^2 + N'^2 + 2H'I' + 2H'M' + 2I'M' + H'N' + I'N' + M'N' - K'M'' - KK'' - N'M',$

$H''^2 + I''^2 + K''^2 + J''^2 + H'^2 + N'^2 + M'^2 + N^2 + M^2 + 2H''I'' + 2H''K'' + 2I''K'' + JJ'' + J'J'' + H'M' + I'M' + I''H'' + K''H'' - HM,$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JM + J'M + J''M + IJ'' - K'M' - N'H'',$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' - H'M - I'M - MM' - K'J'' - KM' - N'H'',$

$H''^2 + I''^2 + K''^2 + N'^2 + N^2 + M^2 + 2H''I'' + 2I''K'' + 2H''K'' + JN' + J'N' + J''N' - IM'' - K'K'' - NM',$

$K'^2 + I'^2 + J'^2 + M''^2 - IK - JJ' - H'I' - I''K'',$

$K'^2 + I'^2 + J'^2 + M''^2 - K'N' - I'M'' - J'K'' - M'M'',$

$N^2 + I''^2 + M''^2 + J''^2 - IM - JJ' - I'M' - I''H'',$

$N^2 + I''^2 + M''^2 + J''^2 - IN' - JM'' - I'K'' - I''M',$

$N^2 + I''^2 + M''^2 + J''^2 - K'M - I'J'' - J'M' - M''H'',$

$N'^2 + M''^2 + K''^2 + M'^2 - KM - J'J'' - H'M' - K''H''.$

are constant. Since every Berwald space is a Landsberg space; these twenty functions are constant in a $S - 3$ like Berwald space. From theorem (3.1) and equation (1.9), function H and $H + I + K + M$ are h−covariant constants in a five-dimensional Berwald space. Therefore in a $S - 3$ like Berwald space, twenty two independent functions H, $H + I + K + M$ and aforesaid twenty functions are h −covariant constants and therefore the main scalars $H, H', H'', I, I', I'', J, J', J'', M, M', M'', K, K', K''$, N and N' are h−covariant constants. Thus, we have

Theorem 4.3. In an S − 3 like five-dimensional Berwald space, all the main scalars are h−covariant constants.

It is clear from (3.2) that if all the main scalars are h −covariants in a Berwald space, then the h−connection vectors h_i, J_i, K_i, h'_i, J'_i and k'_i vanish. This leads to

Theorem 4.4. In an $S - 3$ like five-dimensional Berwald space, the h − connection vectors h_i, J_i, k_i, h'_i, J'_i and k'_i vanish identically.

In view of theorems (4.1), (4.2), (4.3) and (4.4), we can say

Theorem 4.5. In an S−3 like five-dimensional Berwald space satisfying T−condition, all the main scalars are constants and h− and v−connection vectors vanish.

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