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RESEARCH ARTICLE

Theorems on Common Coupled Fixed Point in C*-Algebra Valued Rectangular Parametric Metric Space

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Abstract

In this work, we establish several results concerning common coupled fixed points and coupled coincidence points for mappings that fulfill various contractive conditions within the framework of complete C*-algebra valued rectangular parametric metric space. Furthermore,we demonstrate the applicability of these theorems by proving the existence and uniqueness of solution to a Fredholm nonlinear integral equation.

Keywords: C*-algebra, Coupled fixed point, Rectangular metric space, Parametric metric space, w-compatible.

Introduction

In recent years, the Banach fixed point theorem has been extended by numerous researchers across a variety of settings and frameworks,including b-metric spaces,S_b metric spaces,cone metric spaces,G-metric spaces,partial metric spaces,neutrosophic metric spaces,complex valued metric spaces,bipolar metric spaces,modular metric spaces,etc. In addition,various generalizations of contraction mappings in complete metric spaces have been proposed and developed through different approaches.

The concept of a coupled fixed point was initially introduced by Guo and Lakshmikantham(1987).Bhaskar and Lakshmikantham(2006) further developed the idea of defining the mixed monotonicity property and establishing theorems on coupled fixed points and coupled coincidence points.

The definition of parametric metric spaces was introduced by Hussain *et al*.

Branciari contributed to the field by generalizing the notion of a metric space to a rectangular metric space and proving corresponding fixed point results. Ma *et al*. were the first to present the idea of C*-algebra valued metric spaces and to establish fixed point theorems within this context.

The objective of this paper to establish a common coupled fixed point theorem for mappings satisfying various contractive conditions in the context of C*-algebra valued rectangular parametric metric spaces. We illustrate the applicability of these results by proving the existence and uniqueness of solution to a Fredholm nonlinear integral equation.

Preliminaries

We begin by reviewing some fundamental definitions, notations and results related to C*-algebra.

Let $\mathfrak{A}$ be a unital algebra. An involution on $\mathfrak{A}$ is defined as a conjugate –
linear mapping $\varsigma \longrightarrow \varsigma^*$ that satisfies the following properties,$\forall \varsigma, \omega \in \mathfrak{A}$:

- $(\varsigma\omega)^* = \omega^*\varsigma^*$
- $(\varsigma^*)^* = \varsigma$

An algebra equipped with such an involution is referred to as a *-algebra.If
$\mathfrak{A}$ contains a multiplicative identity element $1_{\mathfrak{A}}$,then the pair $(\mathfrak{A},*)$ is termed a unital *-algebra.

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A Banach *-algebra is a unital *-algebra $(\mathfrak{A},*)$ that is also a complete normed space,where the norm is submultiplicative and satisfies $\|\varsigma^*\| = \|\varsigma\|, ,\forall\varsigma \in \mathfrak{A}$.
Furthermore, if a Banach *-algebra $(\mathfrak{A},*)$ fulfills the condition $\|\varsigma^*\varsigma\| = \|\varsigma\|^2 \ \forall\varsigma \in$ $\mathfrak{A}$, then $\mathfrak{A}$ is known as a C*-algebra.
An element $\varsigma \in \mathfrak{A}$ is called positive if $\varsigma = \varsigma^*$ and its spectrum $\sigma(\varsigma) \subset \mathbb{R}_+$, where $\sigma(\varsigma) = \{\varsigma \in \mathbb{R}: \varsigma 1_{\mathfrak{A}} - \varsigma$ is not invertible}.The set of all positive elements in $\mathfrak{A}$ is denoted by $\mathfrak{A}_+$.
This set $\mathfrak{A}_+$ enables us to introduce a partial order on $\mathfrak{A}$: for $\omega, \varsigma \in \mathfrak{A}$ we write
$\omega \succcurlyeq \varsigma$ if $\omega - \varsigma \in \mathfrak{A}_+$. If $\varsigma \in \mathfrak{A}$ is positive,we denote this by $\varsigma \succcurlyeq 0_{\mathfrak{A}}$,where $0_{\mathfrak{A}}$ is the zero element of $\mathfrak{A}$.Thus, $\mathfrak{A}_+$ denotes can be described as $\{\varsigma \in \mathfrak{A}: \varsigma \succcurlyeq 0_{\mathfrak{A}}\}$ and $(\varsigma^*\varsigma)^{\frac{1}{2}} = |\varsigma|$

Definition 2.1

A mapping $\daleth: \Psi^2 \times (0,\infty) \to \mathfrak{A}$ is said to define C*-algebra valued rectangular parametric metric space on a nonempty set Ψ if the following conditions are satisfied:

- $\daleth(\jmath, \mathfrak{h}, \ell) = 0$ iff $\jmath = \mathfrak{h}, \forall \jmath, \mathfrak{h} \in \Psi \,\&\, \ell > 0$.
- $\daleth(\jmath, \mathfrak{h}, \ell) = \daleth(\mathfrak{h}, \jmath, \ell)$
- $\daleth(\jmath, \mathfrak{h}, \ell) \le \daleth(\jmath, \mathfrak{a}, \ell) + \daleth(\mathfrak{a}, \mathfrak{b}, \ell) + \daleth(\mathfrak{b}, \mathfrak{h}, \ell) \ \forall \jmath, \mathfrak{h}, \mathfrak{a}, \mathfrak{b} \in \Psi \,\&\, \ell > 0$.

Therefore, $(\Psi, \mathfrak{A}, \daleth)$ is a C*-algebra valued rectangular parametric metric space.

Definition 2.2

Consider a C*-algebra valued rectangular parametric metric space $(\Psi,\mathfrak{A},\daleth)$ and a sequence $\{\jmath_n\}$ inΨ. A point $\jmath$ in Ψ is defined as the limit of the sequence $\{\jmath_n\}$, if $\lim_{n\to\infty} \daleth(\jmath, \jmath_n, \ell) = 0$.
In this case, the sequence $\{\jmath_n\}$ is said to converge to $\jmath$.

Definition 2.3

Let $(\Psi,\mathfrak{A},\daleth)$ be a C*-algebra valued rectangular parametric metric space.

- A sequence $\{\jmath_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} \daleth(\jmath_n, \mathfrak{h}_m, \ell) = 0, \forall \ell > 0$.
- $(\boldsymbol{\Psi},\mathfrak{A},\daleth)$ is said to be complete if every Cauchy sequence in $\boldsymbol{\Psi}$ converges.

Definition 2.4

Suppose $\boldsymbol{\Psi}$ is a nonempty set.

- A pair $(\jmath,\mathfrak{h})$ in $\boldsymbol{\Psi} \times \boldsymbol{\Psi}$ is referred to as a coupled fixed point of $\eth: \boldsymbol{\Psi} \times \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ if $\eth(\jmath, \mathfrak{h}) = \jmath$ and $\eth(\mathfrak{h}, \jmath) = \mathfrak{h}$.
- A pair $(\jmath, \mathfrak{h})$ in $\boldsymbol{\Psi} \times \boldsymbol{\Psi}$ is referred to as a coupled coincidence point of $\eth: \boldsymbol{\Psi} \times \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}: \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ if $\eth(\jmath, \mathfrak{h}) = \boldsymbol{\Lambda}\jmath$ and $\eth(\mathfrak{h}, \jmath) = \boldsymbol{\Lambda}\mathfrak{h}$.
- A pair $(\jmath,\mathfrak{h})$ in $\boldsymbol{\Psi} \times \boldsymbol{\Psi}$ is referred to as a common coupled fixed point of $\eth: \boldsymbol{\Psi} \times \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}: \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ if $\eth(\jmath, \mathfrak{h}) = \boldsymbol{\Lambda}\jmath = \jmath$ and $\eth(\mathfrak{h}, \jmath) = \boldsymbol{\Lambda}\mathfrak{h} = \mathfrak{h}$.

Definition 2.5

A pair of mappings $\eth: \Psi \times \Psi \to \Psi$ and $\Lambda: \Psi \to \Psi$ is called w-compatible if, whenever $\Lambda\jmath = \eth(\jmath, \mathfrak{h})$ and $\Lambda\mathfrak{h} = \eth(\mathfrak{h}, \jmath)$, it follows that $\Lambda(\eth(\jmath, \mathfrak{h})) = \eth(\Lambda\jmath, \Lambda\mathfrak{h})$.

Main Results

Theorem 3.1

Let $(\Psi, \mathfrak{A}, \daleth)$ be a complete C*-algebra valued rectangular parametric metric space. Consider two mappings $\eth: \Psi \times \Psi \to \Psi$ and $\Lambda: \Psi \to \Psi$ satisfy the following condition:
$\daleth(\eth(\jmath, \mathfrak{h}), \eth(\mathfrak{m}, \mathfrak{y}), \ell) \le \mathfrak{S}^*\daleth(\Lambda\jmath, \Lambda\mathfrak{m}, \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{y}, \ell)\mathfrak{S}$, (1) for any $\jmath, \mathfrak{h}, \mathfrak{m}, \mathfrak{y} \in \Psi$,
where $\ell > 0$, $\mathfrak{S} \in \mathfrak{A}$ with $\|\mathfrak{S}\| < \frac{1}{\sqrt{2}}$.
Assume further that $\eth(\Psi \times \Psi) \subseteq \Lambda(\Psi)$ and $\Lambda(\Psi)$ is a complete subspace of Ψ,then $\eth$ and Λ have a coupled coincidence point.If the pair $(\eth, \Lambda)$ is w- compatible,then there exists a unique common coupled fixed point in Ψ.

Proof. Let $\jmath_0, \mathfrak{h}_0 \in \Psi$.Define $\Lambda(\jmath_1) = \eth(\jmath_0, \mathfrak{h}_0)$ and $\Lambda(\mathfrak{h}_1) = \eth(\mathfrak{h}_0, \jmath_0)$. By iterating this process,we construct two sequences $\{\jmath_n\}$ and $\{\mathfrak{h}_n\}$ in Ψ such that $\Lambda(\jmath_{n+1}) = \eth(\jmath_n, \mathfrak{h}_n)$ and $\Lambda(\mathfrak{h}_{n+1}) = \eth(\mathfrak{h}_n, \jmath_n)$.Using condition (1) we obtain,
$\daleth(\Lambda\jmath_n, \Lambda\jmath_{n+1}, \ell) = \daleth(\eth(\jmath_{n-1}, \mathfrak{h}_{n-1}), \eth(\jmath_n, \mathfrak{h}_n), \ell)$

$$\begin{aligned} &\le \daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell)\mathfrak{S} \\ &\le \mathfrak{S}^*[\daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell)]\,\mathfrak{S} \end{aligned} \quad (2)$$

In the same way,
$\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell) = \daleth(\eth(\mathfrak{h}_{n-1}, \jmath_{n-1}), \eth(\mathfrak{h}_n, \jmath_n), \ell)$

$$\begin{aligned} &\le \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell)\mathfrak{S} \\ &\le \mathfrak{S}^*[\daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell) + \daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell)]\,\mathfrak{S} \end{aligned} \quad (3)$$

Assume
$\boldsymbol{\Lambda}_n = \daleth(\Lambda\jmath_n, \Lambda\jmath_{n+1}, \ell) + \daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell)$
By combining (2) and (3), it follows that
$\boldsymbol{\Lambda}_n = \daleth(\Lambda\jmath_n, \Lambda\jmath_{n+1}, \ell) + \daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell)$
$\le \mathfrak{S}^*[\daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell)]\mathfrak{S} + \mathfrak{S}^*[\daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n, \ell) + \daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n, \ell)]\,\mathfrak{S}$
$\le (\sqrt{2}\mathfrak{S})\ [\daleth(\Lambda\jmath_{n-1}, \Lambda\jmath_n \jmath_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n-1}, \Lambda\mathfrak{h}_n\mathfrak{h}_n, \ell)](\sqrt{2}\mathfrak{S})$
$\le (\sqrt{2}\mathfrak{S})\ \boldsymbol{\Lambda}_{n-1}\ (\sqrt{2}\mathfrak{S})$
Utilizing the fact that if $\mathfrak{u}\mathfrak{u}, \mathfrak{v}\,\mathfrak{v} \in \mathfrak{A}_{\mathfrak{a}}\mathfrak{A}_{\mathbb{h}}$ and $\mathfrak{u}\mathfrak{u} \le \mathfrak{v}\ \mathfrak{v}$, then $\mathfrak{S}^*\mathfrak{u}\,\mathfrak{S} \le \mathfrak{S}^*\mathfrak{v}\,\mathfrak{S}$,it follows that for any $n \in \mathbb{N}$,

$$0_{\mathfrak{A}} \le \Lambda_n \le \left(\sqrt{2}\mathfrak{S}\right)^* \Lambda_{n-1} \left(\sqrt{2}\mathfrak{S}\right) \le \cdots \le \left(\sqrt{2}\mathfrak{S}\right)^{*n} \Lambda_0 \left(\sqrt{2}\mathfrak{S}\right)^n$$

If $\boldsymbol{\Lambda}_0 = 0_{\mathfrak{A}}$, then the mappings $\boldsymbol{\delta}$ and $\boldsymbol{\Lambda}$ possess a coupled coincidence point $(\jmath_0, \mathfrak{h}_0)$.
With the assumption $\boldsymbol{\Lambda}_0 \le 0_{\mathfrak{A}}$, it can be shown that for any $n, o \in \mathbb{N}$, $\daleth(\boldsymbol{\Lambda}\jmath_{n+o}, \boldsymbol{\Lambda}\jmath_n, \ell) \le \daleth(\boldsymbol{\Lambda}\jmath_{n+o}, \boldsymbol{\Lambda}\jmath_{n+o-1}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-1}, \boldsymbol{\Lambda}\jmath_{n+o-2}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-2}, \boldsymbol{\Lambda}\jmath_n, \ell)$
$\le \daleth(\boldsymbol{\Lambda}\jmath_{n+o}, \boldsymbol{\Lambda}\jmath_{n+o-1}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-1}, \boldsymbol{\Lambda}\jmath_{n+o-2}, \ell)$

$+ [\daleth(\boldsymbol{\Lambda}\jmath_{n+o-2}, \boldsymbol{\Lambda}\jmath_{n+o-3}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-3}, \boldsymbol{\Lambda}\jmath_{n+o-4}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-4}, \boldsymbol{\Lambda}\jmath_n, \ell)]$

$\le \daleth(\boldsymbol{\Lambda}\jmath_{n+o}, \boldsymbol{\Lambda}\jmath_{n+o-1}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-1}, \boldsymbol{\Lambda}\jmath_{n+o-2}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+o-2}, \boldsymbol{\Lambda}\jmath_{n+o-3}, \ell) + \cdots$

$+ \daleth(\boldsymbol{\Lambda}\jmath_{n+3}, \boldsymbol{\Lambda}\jmath_{n+2}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+2}, \boldsymbol{\Lambda}\jmath_{n+1}, \ell) + \daleth(\boldsymbol{\Lambda}\jmath_{n+1}, \boldsymbol{\Lambda}\jmath_n, \ell)$

$\daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell) \leq \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_{n+o-1}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-1}, \Lambda\mathfrak{h}_{n+o-2}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-2}, \Lambda\mathfrak{h}_n, \ell)$

$\leq \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_{n+o-1}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-1}, \Lambda\mathfrak{h}_{n+o-2}, \ell)$

$+ [\daleth(\Lambda\mathfrak{h}_{n+o-2}, \Lambda\mathfrak{h}_{n+o-3}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-3}, \Lambda\mathfrak{h}_{n+o-4}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-4}, \Lambda\mathfrak{h}_n, \ell)]$

$\leq \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_{n+o-1}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-1}, \Lambda\mathfrak{h}_{n+o-2}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-2}, \Lambda\mathfrak{h}_{n+o-3}, \ell) + \cdots$

$+ \daleth(\Lambda\mathfrak{h}_{n+3}, \Lambda\mathfrak{h}_{n+2}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+2}, \Lambda\mathfrak{h}_{n+1}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+1}, \Lambda\mathfrak{h}_n, \ell)$

Thus,

$\daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell)$

$\leq [\daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_{n+o-1}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_{n+o-1}, \ell)]$

$+ [\daleth(\Lambda\mathfrak{j}_{n+o-1}, \Lambda\mathfrak{j}_{n+o-2}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-1}, \Lambda\mathfrak{h}_{n+o-2}, \ell)]$

$+ [\daleth(\Lambda\mathfrak{j}_{n+o-2}, \Lambda\mathfrak{j}_{n+o-3}, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o-2}, \Lambda\mathfrak{h}_{n+o-3}, \ell)] + \cdots$

$+ [\daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+1}, \Lambda\mathfrak{h}_n, \ell)]$

$\leq \Lambda_{n+o-1} + \Lambda_{n+o-2} + \cdots + \Lambda_n$

$\leq (\sqrt{2\mathfrak{S}})^{*n+o-1} \Lambda_0 (\sqrt{2\mathfrak{S}})^{n+o-1} + (\sqrt{2\mathfrak{S}})^{*n+o-2} \Lambda_0 (\sqrt{2\mathfrak{S}})^{n+o-2} + \cdots$

$+ (\sqrt{2\mathfrak{S}})^{*n} \Lambda_0 (\sqrt{2\mathfrak{S}})^{n}$

$\leq \sum_{j=n}^{n+o-1} (\sqrt{2\mathfrak{S}})^{*j} \Lambda_0 (\sqrt{2\mathfrak{S}})^{j}$

From which it follows that,

$$\|\daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell)\| \leq \sum_{j=n}^{n+o-1} \|\sqrt{2\mathfrak{S}}\|^{2j} \Lambda_0$$

$$\leq \sum_{j=n}^{\infty} \|\sqrt{2\mathfrak{S}}\|^{2j} \Lambda_0$$

$$= \frac{\|\sqrt{2\mathfrak{S}}\|^{2j}}{1 - \|\sqrt{2\mathfrak{S}}\|^{2}} \Lambda_0$$

As $\|\mathfrak{S}\| < \frac{1}{\sqrt{2}}$, we obtain

$$\|\daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell)\| \leq \frac{\|\sqrt{2\mathfrak{S}}\|^{2j}}{1 - \|\sqrt{2\mathfrak{S}}\|^{2}} \Lambda_0 \to 0,$$

which is taken together with

$\daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) \leq \daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell)$

and

$\daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell) \leq \daleth(\Lambda\mathfrak{j}_{n+o}, \Lambda\mathfrak{j}_n, \ell) + \daleth(\Lambda\mathfrak{h}_{n+o}, \Lambda\mathfrak{h}_n, \ell)$

implies that $\{\Lambda\mathfrak{j}_n\}$ and $\{\Lambda\mathfrak{h}_n\}$ are Cauchy sequences in $\Lambda(\Psi)$. Because $\{\Lambda\mathfrak{h}_n\}$ is complete,there exist elements $\mathfrak{j}, \mathfrak{h} \in \Psi$ such that $\lim \Lambda\mathfrak{j}_n = \Lambda\mathfrak{j}$ and

$$\lim_{n\to\infty} \Lambda\mathfrak{h}_n = \Lambda.$$

Let us now prove that $(\mathfrak{j}, \mathfrak{h}) = \Lambda\mathfrak{j}$ and $\delta(\mathfrak{h}, \mathfrak{j}) = \Lambda\mathfrak{h}$. For this,

$\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) \leq \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}_{n+1}, \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)$

$\leq \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \delta(\mathfrak{j}_n, \mathfrak{h}_n), \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)$

$\leq \mathfrak{S}^*\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}, \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}, \ell)\mathfrak{S} + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)$

As $n \to \infty$, we have $\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) = 0_{\mathfrak{A}}$, which implies that $\delta(\mathfrak{j}, \mathfrak{h}) = \Lambda\mathfrak{j}$.In a similar manner, $\delta(\mathfrak{h}, \mathfrak{j}) = \Lambda\mathfrak{h}$. Thus, δ and Λ have a coupled coincidence point $(\mathfrak{j}, \mathfrak{h})$.

Suppose $(\mathfrak{j}', \mathfrak{h}')$ is another coupled coincidence point of δ and Λ. Then,

$\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) = \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \delta(\mathfrak{j}', \mathfrak{h}'), \ell)$

$\leq \mathfrak{S}^*\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell)\mathfrak{S}$ $\daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell) = \daleth(\delta(\mathfrak{h}, \mathfrak{j}), \delta(\mathfrak{h}', \mathfrak{j}'), \ell)$

$\leq \mathfrak{S}^*\daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell)\mathfrak{S}$

As a result,

$$\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) + \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell) \leq (\sqrt{2\mathfrak{S}})^*(\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) + \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell))(\sqrt{2\mathfrak{S}})$$

from which it follows that

$\|\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) + \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell)\| \leq \|\sqrt{2\mathfrak{S}}\| \; \|\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) + \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell)\|$.

Given that $\|\sqrt{2\mathfrak{S}}\| < 1$,it follows that $\|\daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}', \ell) + \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}', \ell)\| = 0$.Thus, we conclude $\Lambda\mathfrak{j} = \Lambda\mathfrak{j}'$ and $\Lambda\mathfrak{h} = \Lambda\mathfrak{h}'$. Following a similar argument,one can also show that $\Lambda\mathfrak{j} = \Lambda\mathfrak{h}'$ and $\Lambda\mathfrak{h} = \Lambda\mathfrak{j}'$. Then δ and Λ have a unique coupled coincidence point $(\Lambda\mathfrak{j}, \Lambda\mathfrak{j})$.

Now,let $\mathfrak{x} = \Lambda\mathfrak{j}$.Then $\mathfrak{x} = \Lambda\mathfrak{j} = \delta(\mathfrak{j}, \mathfrak{j})$. Since δ and Λ are w-compatible,we have

$\Lambda\mathfrak{x} = \Lambda(\Lambda\mathfrak{j}) = \Lambda(\delta(\mathfrak{j}, \mathfrak{j})) = \delta(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}) = \delta(\mathfrak{x}, \mathfrak{x})$.

Therefore, δ and Λ possess a coupled coincidence point $(\Lambda\mathfrak{x}, \Lambda\mathfrak{x})$.Since, $\Lambda\mathfrak{x} = \Lambda\mathfrak{j}$, we have $\mathfrak{x} = \Lambda\mathfrak{j} = (\mathfrak{j}, \mathfrak{j})$. Consequently, δ and Λ have a unique common coupled fixed point $(\mathfrak{x}, \mathfrak{x})$.

Corollary 3.2

Let $(\Psi, \mathfrak{A}, \daleth)$ be a complete C*-algebra valued rectangular parametric metric space. Assume there is a mapping $\delta: \Psi \times \Psi \to \Psi$ that satisfies the condition

$$\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \delta(\mathfrak{m}, \mathfrak{y}), \ell) \leq \mathfrak{S}^*\daleth(\mathfrak{j}, \mathfrak{m}, \ell)\mathfrak{S} + \mathfrak{S}^*\daleth(\mathfrak{h}, \mathfrak{y}, \ell)\mathfrak{S}, \quad (4)$$

for any $\mathfrak{j}, \mathfrak{h}, \mathfrak{m}, \mathfrak{y} \in \Psi$,

where $\ell > 0$, $\mathfrak{S} \in \mathfrak{A}$ with $\|\mathfrak{S}\| < \frac{1}{}$.

Under these assumptions, δ possesses a unique coupled fixed point in Ψ.

Lemma 3.3[14]

Suppose that $\mathfrak{A}$ is a unital C*-algebra with a unit $1_{\mathfrak{A}}$.

- If $\mathfrak{j} \in \mathfrak{A}_+$ with $\|\mathfrak{j}\| < \frac{1}{2}$, then $1_{\mathfrak{A}} - \mathfrak{j}$ is invertible
- If $\mathfrak{j}, \mathfrak{h} \in \mathfrak{A}_+$ and $\mathfrak{j}\mathfrak{h} = \mathfrak{h}\mathfrak{j}$, then $0_{\mathfrak{A}} \leq \mathfrak{j}\mathfrak{h}$
- If $\mathfrak{j}, \mathfrak{h} \in \mathfrak{A}_{\mathbb{h}}$ and $\mathfrak{k} \in \mathfrak{A}_+'$ then $\mathfrak{j} \leq \mathfrak{h}$ deduces $\mathfrak{k}\mathfrak{j} \leq \mathfrak{k}\mathfrak{h}$, where $\mathfrak{A}' = \mathfrak{A}_+ \cap \mathfrak{A}'$.

Theorem 3.4

Let $(\Psi, \mathfrak{A}, \daleth)$ is a complete C*-algebra valued rectangular parametric metric space. Consider two mappings $\delta: \Psi \times \Psi \to \Psi$ and $\Lambda: \Psi \to \Psi$ such that

$\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \delta(\mathfrak{m}, \mathfrak{y}), \ell) \leq x \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) + y \daleth(\delta(\mathfrak{m}, \mathfrak{y}), \Lambda\mathfrak{m}, \ell)$, (5)

for any $\mathfrak{j}, \mathfrak{h}, \mathfrak{m}, \mathfrak{y} \in \Psi$,

where $\ell > 0$, $x, y \in \mathfrak{A}_+'$ with $\|x\| + \|y\| < 1$.

Assume further that $(\Psi \times \Psi) \subseteq \Lambda(\Psi)$ and $\Lambda(\Psi)$ is a complete subspace of Ψ, then δ and Λ have a coupled coincidence point.If the pair (δ, Λ) is w- compatible,then exists a unique common coupled fixed point in Ψ.

Proof.

As in the proof of Theorem 3.1,construct the sequences $\{\mathfrak{j}_n\}$ and $\{\mathfrak{h}_n\}$ in Ψ with $\Lambda(\mathfrak{j}_{n+1}) = \delta(\mathfrak{j}_n, \mathfrak{h}_n)$ and $\Lambda(\mathfrak{h}_{n+1}) = \delta(\mathfrak{h}_n, \mathfrak{j}_n)$.Then using (5),

$$\begin{aligned}\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) &= \daleth(\delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \delta(\mathfrak{j}_n, \mathfrak{h}_n), \ell)\\ &\leq x \daleth(\delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \Lambda\mathfrak{j}_{n-1}, \ell) + y \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \Lambda\mathfrak{j}_n, \ell)\\ &= x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell) + y \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell)\end{aligned}$$

Thus,

$$(1_{\mathfrak{A}} - y)\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)$$

In the same way,

$$(1_{\mathfrak{A}} - y)\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell) \leq x \daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n-1}, \ell)$$

Given that $x, y \in \mathfrak{A}_+'$ and $\|x\| + \|y\| < 1$,it follows that $(1_{\mathfrak{A}} - y)$ is invertible and $(1_{\mathfrak{A}} - y)^{-1}x \in \mathfrak{A}_+'$ .Hence,

$$\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq (1_{\mathfrak{A}} - y)^{-1}x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)$$

$$\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell) \leq (1_{\mathfrak{A}} - y)^{-1}x \daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n-1}, \ell)$$

Thus,

$$\|\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell)\| \leq \|(1_{\mathfrak{A}} - y)^{-1}x\|\|\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)\|,$$

$$\|\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n+1}, \ell)\| \leq \|(1_{\mathfrak{A}} - y)^{-1}x\|\|\daleth(\Lambda\mathfrak{h}_n, \Lambda\mathfrak{h}_{n-1}, \ell)\|,$$

It follows from the fact that

$$\|(1_{\mathfrak{A}} - y)^{-1}x\| \leq \|(1_{\mathfrak{A}} - y)^{-1}\|\|x\| \leq \sum_{k=0}^{\infty}\|y\|^k \|x\| = \frac{\|x\|}{1 - \|y\|} < 1.$$

Thus,the sequences $\{\Lambda\mathfrak{j}_n\}$ and $\{\Lambda\mathfrak{h}_n\}$ are Cauchy sequences in $\Lambda(\Psi)$. Since Ψ is complete,there exist $\mathfrak{j}, \mathfrak{h} \in \Psi$ such that $\lim \Lambda\mathfrak{j}_n = \Lambda\mathfrak{j}$ and

$\lim_{n\to\infty} \Lambda\mathfrak{h}_n = \Lambda\mathfrak{h}$.

Since,

$$\begin{aligned}\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) &\leq \daleth(\Lambda\mathfrak{j}_{n+1}, \delta(\mathfrak{j}, \mathfrak{h}), \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)\\ &= \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \delta(\mathfrak{j}, \mathfrak{h}), \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)\\ &\leq x \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \Lambda\mathfrak{j}_n, \ell) + y \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)\\ &\leq x \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) + y \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) + \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)\end{aligned}$$

which leads to

$$\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) \leq (1 - y)^{-1}x \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) + (1 - y)^{-1}\daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}, \ell)$$

Like above

$$\|(1_{\mathfrak{A}} - y)^{-1}x\| \leq \|(1_{\mathfrak{A}} - y)^{-1}\|\|x\| \leq \sum_{k=0}^{\infty}\|y\|^k \|x\| = \frac{\|x\|}{1 - \|y\|} < 1$$

As a result, $\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell) = 0_{\mathfrak{A}}$ or equivalently $\delta(\mathfrak{j}, \mathfrak{h}) = \Lambda\mathfrak{j}$. In the same way,

$\delta(\mathfrak{h}, \mathfrak{j}) = \Lambda\mathfrak{h}$. Now, suppose $(\mathfrak{j}', \mathfrak{h}')$ is another coupled coincidence point of δ and Λ. In that case,

$$\begin{aligned}\daleth(\Lambda\mathfrak{j}', \Lambda\mathfrak{j}, \ell) &\leq \daleth(\delta(\mathfrak{j}', \mathfrak{h}'), \delta(\mathfrak{j}, \mathfrak{h}), \ell)\\ &\leq x \daleth(\delta(\mathfrak{j}', \mathfrak{h}'), \Lambda\mathfrak{j}', \ell) + y \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{j}, \ell)\\ &= x \daleth(\Lambda\mathfrak{j}', \Lambda\mathfrak{j}', \ell) + y \daleth(\Lambda\mathfrak{j}, \Lambda\mathfrak{j}, \ell) = 0_{\mathfrak{A}}.\end{aligned}$$

Hence, $\daleth(\Lambda\mathfrak{j}', \Lambda\mathfrak{j}, \ell) = 0_{\mathfrak{A}}$, and then $\Lambda\mathfrak{j}' = \Lambda\mathfrak{j}$. In the same way,

$$\begin{aligned}\daleth(\Lambda\mathfrak{h}', \Lambda\mathfrak{h}, \ell) &\leq \daleth(\delta(\mathfrak{h}', \mathfrak{j}'), \delta(\mathfrak{h}, \mathfrak{j}), \ell)\\ &\leq x \daleth(\delta(\mathfrak{h}', \mathfrak{j}'), \Lambda\mathfrak{h}', \ell) + y \daleth(\delta(\mathfrak{h}, \mathfrak{j}), \Lambda\mathfrak{h}, \ell)\\ &= x \daleth(\Lambda\mathfrak{h}', \Lambda\mathfrak{h}', \ell) + y \daleth(\Lambda\mathfrak{h}, \Lambda\mathfrak{h}, \ell) = 0_{\mathfrak{A}}.\end{aligned}$$

Hence, $\daleth(\Lambda\mathfrak{h}', \Lambda\mathfrak{h}, \ell) = 0_{\mathfrak{A}}$, and then $\Lambda\mathfrak{h}' = \Lambda\mathfrak{h}$. Similarly, we have $\Lambda\mathfrak{j}' = \Lambda\mathfrak{h}$ and $\Lambda\mathfrak{h}' = \Lambda\mathfrak{j}$. Thus, δ and Λ have a unique coupled coincidence point $(\Lambda\mathfrak{j}, \Lambda\mathfrak{j})$.Furthermore, it can be demonstrated that δ and Λ possess a unique common coupled fixed point.

Theorem 3.5

Let $(\Psi, \mathfrak{A}, \daleth)$ is a complete C*-algebra valued rectangular parametric metric space. Consider two mappings $\delta: \Psi \times \Psi \to \Psi$ and $\Lambda: \Psi \to \Psi$ satisfy the following condition:

$\daleth(\delta(\mathfrak{j}, \mathfrak{h}), \delta(\mathfrak{m}, \mathfrak{y}), \ell) \leq x \daleth(\delta(\mathfrak{j}, \mathfrak{h}), \Lambda\mathfrak{m}, \ell) + y \daleth(\delta(\mathfrak{m}, \mathfrak{y}), \Lambda\mathfrak{j}, \ell)$, (6)

for any $\mathfrak{j}, \mathfrak{h}, \mathfrak{m}, \mathfrak{y} \in \Psi$,

where $\ell > 0$, $x, y \in \mathfrak{A}_+'$ with $\|x\| + \|y\| < 1$.

Assume further that $(\Psi \times \Psi) \subseteq \Lambda(\Psi)$ and $\Lambda(\Psi)$ is a complete subspace of Ψ,then δ and Λ have a coupled coincidence point.If the pair (δ, Λ) is w- compatible,then there exists a unique common coupled fixed point in Ψ.

Proof.

As in the proof of Theorem 3.1, construct two sequences $\{_n\}$ and $\{\mathfrak{h}_n\}$ in Ψ with $\Lambda(\mathfrak{j}_{n+1}) = \delta(\mathfrak{j}_n, \mathfrak{h}_n)$ and $\Lambda(\mathfrak{h}_{n+1}) = \delta(\mathfrak{h}_n, \mathfrak{j}_n)$.Using equation(6),

$$\begin{aligned}\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) &= \daleth(\delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \delta(\mathfrak{j}_n, \mathfrak{h}_n), \ell)\\ &\leq x \daleth(\delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \Lambda\mathfrak{j}_n, \ell) + y \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \Lambda\mathfrak{j}_{n-1}, \ell)\\ &\leq x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_n, \ell) + y \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_{n-1}, \ell)\\ &= y \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_{n-1}, \ell)\\ &\leq y \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) + y \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_n, \ell) + y \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)\end{aligned}$$

which implies

$$(1_{\mathfrak{A}} - y)\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq y \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell) \quad (7)$$

Due to the symmetry in (6) , we obtain

$$\begin{aligned}\daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) &= \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \ell)\\ &\leq x \daleth(\delta(\mathfrak{j}_n, \mathfrak{h}_n), \Lambda\mathfrak{j}_{n-1}, \ell) + y \daleth(\delta(\mathfrak{j}_{n-1}, \mathfrak{h}_{n-1}), \Lambda\mathfrak{j}_n, \ell)\\ &\leq x \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_{n-1}, \ell) + y \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_n, \ell)\\ &= x \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_{n-1}, \ell)\\ &\leq x \daleth(\Lambda\mathfrak{j}_{n+1}, \Lambda\mathfrak{j}_n, \ell) + x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_n, \ell) + x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)\end{aligned}$$

which implies,

$$(1_{\mathfrak{A}} - x)\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq x \daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell) \quad (8)$$

Using (7) and (8), we deduce that

$$(1_{\mathfrak{A}} - \frac{x+y}{2})\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq \frac{x+y}{2}\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)$$

Given that $x, y \in \mathfrak{A}_+'$ and $\|x + y\| \leq \|x\| + \|y\| < 1$, it follows that $(1_{\mathfrak{A}} - \frac{x+}{2})^{-1} \in \mathfrak{A}_+'$.Combined with Lemma 3.3(3), it leads to

$$\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n+1}, \ell) \leq (1_{\mathfrak{A}} - \frac{x+y}{2})^{-1}\frac{x+y}{2}\daleth(\Lambda\mathfrak{j}_n, \Lambda\mathfrak{j}_{n-1}, \ell)$$

Define $\mathfrak{T} = (1_{\mathfrak{A}} - \frac{x+y}{2})^{-1}\frac{x+y}{2}$. Then,

$$\|\mathfrak{T}\| = \left\|(1_{\mathfrak{A}} - \frac{x+y}{2})^{-1}\frac{x+y}{2}\right\| < 1$$

Applying the same reasoning as in Theorem 3.4, we find that $\{\Lambda\mathfrak{j}_n\}$ is a Cauchy sequence in $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$. Likewise,it can be shown that $\{\boldsymbol{\Lambda}\,\mathfrak{h}_n\}$ are Cauchy sequences in $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$. Since $(\boldsymbol{\Psi})$ is complete,there exist $\mathfrak{j}, \mathfrak{h} \in \boldsymbol{\Psi}$ such that

$\lim_{n\to\infty} \boldsymbol{\Lambda}\mathfrak{j}_n = \boldsymbol{\Lambda}\mathfrak{j}$ and $\lim_{n\to\infty} \boldsymbol{\Lambda}\mathfrak{h}_n = \boldsymbol{\Lambda}\mathfrak{h}$.

Next, we demonstrate that $\eth(\mathfrak{j}, \mathfrak{h}) = \Lambda\mathfrak{j}$ and $\boldsymbol{\delta}(\mathfrak{h}, \mathfrak{j}) = \boldsymbol{\Lambda}\mathfrak{h}$. $\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}, \ell) \leq \daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \ell) + \daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)$

$= \daleth(\boldsymbol{\delta}(\mathfrak{j}_n, \mathfrak{h}_n), \boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \ell) + \daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)$

$\leq x\,\daleth(\boldsymbol{\delta}(\mathfrak{j}_n, \mathfrak{h}_n), \boldsymbol{\Lambda}\mathfrak{j}, \ell) + y\,\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}_n, \ell) + \daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)$

$\leq x\,\daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell) + y\,\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}_n, \ell) + \daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)$

which implies

$\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}, \ell)\| \leq \|x\|\|\daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)\| + \|y\|\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}_n, \ell)\| + \|\daleth(\boldsymbol{\Lambda}\mathfrak{j}_{n+1}, \boldsymbol{\Lambda}\mathfrak{j}, \ell)\|$

By virtue of the continuity of both the metric and the norm, it follows that

$\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}, \ell)\| \leq \|y\|\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}, \ell)\|$.

Since $\|y\| < 1$,it implies that $\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\Lambda}\mathfrak{j}, \ell)\| = 0$, and consequently $\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}) = \boldsymbol{\Lambda}\mathfrak{j}$. By analogous argument,we can obtain $\boldsymbol{\delta}(\mathfrak{h}, \mathfrak{j}) = \boldsymbol{\Lambda}\mathfrak{h}$.Thus, $(\mathfrak{j}, \mathfrak{h})$ constitutes a coupled coincidence point of $\boldsymbol{\delta}$ and $\boldsymbol{\Lambda}$.By applying the same argument as in Theorem 3.4, we can conclude that $\boldsymbol{\delta}$ and $\boldsymbol{\Lambda}$ have unique common coupled fixed point in $\boldsymbol{\Psi}$.

Application

In this section, we illustrate the applicability of the coupled fixed point results established in Section 3 by demonstrating the existence and uniqueness of a solution for a system of integral equations. Specifically,we examine a coupled system of two nonlinear integral equations given by:

$$\mathfrak{j}(t) = \mathfrak{g}(t) + \int_{\mathcal{E}} \wp(t, \lambda, \mathfrak{j}(\lambda), \mathfrak{h}(\lambda))\, d\lambda \qquad , t, \lambda \in \mathcal{E}$$

$$\mathfrak{h}(t) = \mathfrak{g}(t) + \int_{\mathcal{E}} \wp(t, \lambda, \mathfrak{h}(\lambda), \mathfrak{j}(\lambda))\, d\lambda \qquad , t, \lambda \in \mathcal{E}$$

where E is a measurable set. Suppose (for all $\mathfrak{j}, \mathfrak{h} \in \boldsymbol{\Psi}$)

- $\wp: \mathcal{E} \times \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{g} \in \mathcal{L}^\infty(\mathcal{E})$.
- There exists a continuous function p: E × E → R and $\mu \in (0,1)$, such that

$$|\wp(t, \lambda, \mathfrak{j}(\lambda), \mathfrak{h}(\lambda)) - \wp(t, \lambda, \mathfrak{m}(\lambda), \mathfrak{y}(\lambda))|$$
$$\leq \mu|p(t,\lambda)|(|\mathfrak{j}(\lambda) - \mathfrak{m}(\lambda)| + |\mathfrak{h}(\lambda) - \mathfrak{y}(\lambda)| + I - \mu^{-1}I)$$

for all $t, \lambda \in \mathcal{E}$.

$$\sup_{t \in \mathcal{E}} \int_{\mathcal{E}} |p(t, \lambda)| d\lambda \leq 1.$$

Then the integral equation has a unique solution in $\boldsymbol{\Psi}$.

Proof.

Let $\xi = \mathcal{L}^\infty(\mathcal{E})$ denote the set of essentially bounded measurable functions on $\mathcal{E}$ and $\mathfrak{B}(\mathcal{L}^2(\mathcal{E}))$ represent the set of bounded linear operators on a Hilbert space $\mathcal{L}^2(\mathcal{E})$. Define the mapping $\varphi: \xi \times \xi \to (\mathcal{L}^2(\mathcal{E}))$ by $\varphi(\mathfrak{j}, \mathfrak{h}, \ell) = \pi_{\ell|\mathfrak{j}-\mathfrak{h}|^r}$ where $\pi_a: \mathfrak{K} \to \mathfrak{K}$ is multiplication operator given by $\pi(\mathfrak{z}) = a.\mathfrak{z}$ for $\mathfrak{z} \in \mathfrak{K}$ and $r > 1$.

Define $\gamma: \boldsymbol{\Psi} \times \boldsymbol{\Psi} \to \boldsymbol{\Psi}$ by

$$\gamma(\mathfrak{j}, \mathfrak{h})(t) = \mathfrak{g}(t) + \int_{\mathcal{E}} \wp(t, \lambda, \mathfrak{j}(\lambda), \mathfrak{h}(\lambda))\, d\lambda \ \ \forall t, \lambda \in \mathcal{E}$$

Define $\omega = \mu I$, then $\omega \in \mathfrak{A}$ and $\|\omega\| = \mu < 1$. For any $q \in \mathcal{L}^2(\mathcal{E})$, we have

$$\|\daleth(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}), \boldsymbol{\delta}(\mathfrak{m}, \mathfrak{y}), \ell)\| = \sup_{\|\mathfrak{q}\|=1}(\pi_{\ell|\boldsymbol{\delta}(\mathfrak{j},\mathfrak{h}) - \boldsymbol{\delta}(\mathfrak{m},\mathfrak{y})|^r + I}\mathfrak{q}, \mathfrak{q})$$

$$= \ell \sup_{\|\mathfrak{q}\|=1} \int_{\mathcal{E}} |(\boldsymbol{\delta}(\mathfrak{j}, \mathfrak{h}) - \boldsymbol{\delta}(\mathfrak{m}, \mathfrak{y})|^r + I)\mathfrak{q}(t)\bar{\mathfrak{q}}(t) d\, t$$

$$\leq \ell \sup_{\|\mathfrak{q}\|=1} \int_{\mathcal{E}}[\int_{\mathcal{E}} |\wp(t, \lambda, \mathfrak{j}(\lambda), \mathfrak{h}(\lambda)) - \wp(t, \lambda, \mathfrak{m}(\lambda), \mathfrak{y}(\lambda))|d\lambda]^r|\mathfrak{q}(t)|^2 dt + \sup_{\|\mathfrak{q}\|=1} \int_{\mathcal{E}} |(t)|^2 dtI$$

$$\leq \ell \sup_{\|\mathfrak{q}\|=1} \int_{\mathcal{E}}[\int_{\mathcal{E}} |\mu|p(t, \lambda)|(|\mathfrak{j}(\lambda) - \mathfrak{m}(\lambda)| + |\mathfrak{h}(\lambda) - \mathfrak{y}(\lambda)| + I - \mu^{-1}I)|d\lambda]^r|\mathfrak{q}(t)|^2 dt + I$$

$$\leq \ell\, \mu^r \sup_{\|\mathfrak{q}\|=1} \int_{\mathcal{E}}[\int_{\mathcal{E}} |p(t, \lambda)|d\lambda]^r|\mathfrak{q}(t)|^2 dt(\|\mathfrak{j} - \mathfrak{m}\|^r_{\infty} + \|\mathfrak{h} - \mathfrak{y}\|^r_{\infty})$$

$$\leq \mu \sup_{t\in\mathcal{E}} \int_{\mathcal{E}} |p(t, \lambda)|d\lambda \int_{\mathcal{E}} |(t)|^2 dt(\ell[\|\mathfrak{j} - \mathfrak{m}\|^r_{\infty} + \|\mathfrak{h} - \mathfrak{y}\|^r_{\infty}])$$

$$\leq \mu(\ell[\|\mathfrak{j} - \mathfrak{m}\|^r_{\infty} + \|\mathfrak{h} - \mathfrak{y}\|^r_{\infty}])$$

$$= \|\omega\|[\|\daleth(\mathfrak{j}, \mathfrak{m}, \ell)\| + \|\daleth(\mathfrak{h}, \mathfrak{y}, \ell)\|].$$

Consequently,all the hypotheses of Corollary 3.2 are satisfied. It follows that the integral equation admits a unique solution.

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