THE SCIENTIFIC TEMPER (2025) Vol. 16 (12): 5328-5336
E-ISSN: 2231-6396, ISSN: 0976-8653
Doi: 10.58414/SCIENTIFICTEMPER.2025.16.12.20
https://scientifictemper.com/

RESEARCH ARTICLE

Fixed Point and Coupled Fixed Point Theorems on Modular Bipolar Metric Space with an Application to Integral Equations

Maria Merlin S[1], Leema Maria Prakasam A[2*]

Abstract

We introduce the concept of modular bipolar metric space and prove fixed point and coupled fixed point theorems on modular bipolar metric space using covariant and contravariant conditions. We provide an application to an integral equation.

Keywords: cauchy bisequence, contravariant map, covariant map, modular bipolar metric.

Introduction

Fréchet introduced the theory of metric spaces. Numerous problems are solved with the existence of unique solutions by using the theory of fixed point. In literature, there are many kinds of metric spaces such as partial, rectangular, cone, b-metric, etc.Mutlu and Gurdal (2016) introduced the concept of Bipolar metric space and proved the existence of fixed point results for covariant and contravariant contractions. Also, they investigated some fixed point and coupled fixed point results on this space.Chistyakov introduced the notion of modular metric spaces generated by F-modular and develop the theory of this spaces. Also, he defined the notion of a modular on an arbitrary set and develop the theory of metric spaces generated by modular called the modular metric spaces. The idea of a coupled fixed point was initially proposed by Guo and Lakshmikantham(1987).Bhaskar and Lakshmikantham(2006) looked into a few coupled fixed point theorems for mappings and first proposed the idea of a mixed monotone property. Many authors were able to derive numerous fixed point and coupled fixed point and coupled coincidence theorems as a result.In this article, we proposed a new concept "Modular Bipolar metric space". We study and prove the existence of fixed point and coupled fixed point theorems. We provide an application in integral equations to validate our results.

[1]Research Scholar, PG and Research Department of Mathematics, Holy Cross College (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli 620002, Tamil Nadu, India.

[2]Assistant Professor, PG and Research Department of Mathematics, Holy Cross College (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli 620002, Tamil Nadu, India.

***Corresponding Author:** Leema Maria Prakasam A, Research Scholar, PG and Research Department of Mathematics, Holy Cross College (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli 620002, Tamil Nadu, India, E-Mail: leemamaria15@gmail.com

How to cite this article: Merlin, M.S., Prakasam L.M.A. (2025). Fixed Point and Coupled Fixed Point Theorems on Modular Bipolar Metric Space with an Application to Integral Equations. The Scientific Temper, **16**(12):5328-5336.

Doi: 10.58414/SCIENTIFICTEMPER.2025.16.12.20

Source of support: Nil

Conflict of interest: None.

Preliminaries

Definition 1. [7] A function $w_\varphi : (0,\infty) \times \mathfrak{M} \times \mathfrak{M} \to [0,\infty]$ is said to be a (metric) modular on $\mathfrak{M}$ if it satisfies the following conditions:

- $\mathfrak{p}=\mathfrak{q}$ if and only if $w_\varphi(\mathfrak{p},\mathfrak{q})=0$ for all $\varphi>0$.
- $w_\varphi(\mathfrak{p},\mathfrak{q})=w_\varphi(\mathfrak{q},\mathfrak{p})$.
- $w_{\varphi+\tau}(\mathfrak{p},\mathfrak{r}) \le w_\varphi(\mathfrak{p},\mathfrak{q})+w_\tau(\mathfrak{q},\mathfrak{r})$ for all $\varphi,\tau>0$ and all $\mathfrak{p},\mathfrak{q},\mathfrak{r}\in\mathfrak{M}$.

Remark. [18] A modular w_φ on a set $\mathfrak{M}$, the function $0<\lambda\mapsto w_\varphi(\mathfrak{p},\mathfrak{q}) \in[0,\infty]$ for all $\mathfrak{p},\mathfrak{q} \in \mathfrak{M}$, is a non-increasing on $(0, \infty)$.

Definition 2. [12] Let $\mathfrak{M}$ and *B* be nonempty sets and let $w_\varphi : \mathfrak{M} \times \mathfrak{V} \to \mathbb{R}^+$ be a function, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Consider the following properties:

(B1) If $w_\varphi(\mathfrak{p},\mathfrak{q})=0$, then $\mathfrak{p}=\mathfrak{q}$ for all $(\mathfrak{p},\mathfrak{q})\in\mathfrak{M}\times\mathfrak{V}$.

(B2) If $\mathfrak{p}=\mathfrak{q}$, then $w_\varphi(\mathfrak{p},\mathfrak{q})=0$ for all $(\mathfrak{p},\mathfrak{q})\in\mathfrak{M}\times\mathfrak{V}$.

Received: 21/11/2025 **Accepted:** 22/12/2025 **Published:** 25/12/2025

(B3) $w_\varphi(\mathfrak{p},\mathfrak{q}) = w_\varphi(\mathfrak{q},\mathfrak{p})$ for all $\mathfrak{p},\mathfrak{q} \in \mathfrak{M} \cap \mathfrak{V}$.

(B4) $w_\varphi(\mathfrak{p}_1,\mathfrak{q}_2) \le w_\varphi(\mathfrak{p}_1,\mathfrak{q}_1) + w_\varphi(\mathfrak{p}_2,\mathfrak{q}_1) + w_\varphi(\mathfrak{p}_2,\mathfrak{q}_2)$,
for all $\mathfrak{p}_1,\mathfrak{p}_2 \in \mathfrak{M}$ and $\mathfrak{q}_1,\mathfrak{q}_2 \in \mathfrak{V}$.
Then:

- If (B2) and (B3) hold, then w_φ is called a bipolar pseudo-semimetric on the pair ($\mathfrak{M}$, $\mathfrak{V}$).
- If w_φ is a bipolar pseudo-semimetric satisfying (B4), it is called a *bipolar pseudo-metric*.
- A bipolar pseudo-metric w_φ satisfying (B1) is called a bipolar metric.

Definition 3. A function $w_\varphi : (0,\infty) \times \mathfrak{M} \times \mathfrak{V} \to [0,\infty]$ is said to be modular bipolar metric space on $\mathfrak{M} \times \mathfrak{V}$ if it satisfies the following conditions:

- $w_\varphi(\mathfrak{p},\mathfrak{q}) = 0$ iff $\mathfrak{p} = \mathfrak{q}$ for all $\varphi > 0, (\mathfrak{p},\mathfrak{q}) \in \mathfrak{M} \times \mathfrak{V}$.
- $w_\varphi(\mathfrak{p},\mathfrak{q}) = w_\varphi(\mathfrak{q},\mathfrak{p})$ for all $\varphi > 0, (\mathfrak{p},\mathfrak{q}) \in \mathfrak{M} \cap \mathfrak{V}$.

$$w_{\varphi+\delta+\tau}(\mathfrak{p}_1,\mathfrak{q}_2) \le w_\varphi(\mathfrak{p}_1,\mathfrak{q}_1) + w_\delta(\mathfrak{p}_2,\mathfrak{q}_1) + w_\tau(\mathfrak{p}_2,\mathfrak{q}_2),$$

for all $\varphi, \delta, \tau > 0$, for all $\mathfrak{p}_1,\mathfrak{p}_2 \in \mathfrak{M}, \mathfrak{q}_1,\mathfrak{q}_2 \in \mathfrak{V}$.

Definition 4. Let $(\mathfrak{M}_1,\mathfrak{V}_1)$ and $(\mathfrak{M}_2,\mathfrak{V}_2)$ be pairs of sets.

Let $\xi : \mathfrak{M}_1 \cup \mathfrak{V}_1 \to \mathfrak{M}_2 \cup \mathfrak{V}_2$.

- Covariant map : If $\xi(\mathfrak{M}_1) \subseteq \mathfrak{M}_2$ and $\xi(\mathfrak{V}_1) \subseteq \mathfrak{V}_2$, then $\xi : (\mathfrak{M}_1,\mathfrak{V}_1) \rightrightarrows (\mathfrak{M}_2,\mathfrak{V}_2)$.
- Contravariant map :If $\xi(\mathfrak{M}_1) \subseteq \mathfrak{V}_2$ and $\xi(\mathfrak{V}_1) \subseteq \mathfrak{M}_2$, then $\xi : (\mathfrak{M}_1, \mathfrak{V}_1) \nearrow\searrow (\mathfrak{M}_2, \mathfrak{V}_2)$.
- If w_{φ_1} and w_{φ_2} are modular bipolar metrics on $(\mathfrak{M}_1,\mathfrak{V}_1)$ and $(\mathfrak{M}_2,\mathfrak{V}_2)$

respectively, we write $\xi : (\mathfrak{M},\mathfrak{V},w_{\varphi_1}) \rightrightarrows (\mathfrak{M},\mathfrak{V},w_{\varphi_2})$ and $\xi : (\mathfrak{M},\mathfrak{V},w_{\varphi_1}) \nearrow\searrow (\mathfrak{M},\mathfrak{V},w_{\varphi_2})$.

Definition 5. Let ($\mathfrak{M}$, $\mathfrak{V}$, w_φ) be a modular bipolar metric space.

(a) A point $\mathfrak{u} \in \mathfrak{M} \cup \mathfrak{V}$ is called:

- a left point i f $\mathfrak{u} \in \mathfrak{M}$,
- a right point if $\mathfrak{u} \in \mathfrak{V}$,
- a central point if $\mathfrak{u} \in \mathfrak{M} \cap \mathfrak{V}$.

(b) A sequence ($\mathfrak{p}_n$) in $\mathfrak{M}$ is a left sequence; ($\mathfrak{q}_n$) in $\mathfrak{V}$ is a right sequence.

(c) A sequence ($\mathfrak{u}_n$) is said to converge to $\mathfrak{u}$ if either:

- ($\mathfrak{u}_n$) is a left sequence, $\mathfrak{u}$ is a right point, and $\lim_{n\to\infty} w_\varphi(\mathfrak{u}_n, \mathfrak{u}) = 0$, or
- ($\mathfrak{u}_n$) is a right sequence, $\mathfrak{u}$ is a left point, and $\lim_{n\to\infty} w_\varphi(\mathfrak{u}, \mathfrak{u}_n) = 0$.

(d) A pair ($\mathfrak{p}_n$, $\mathfrak{q}_n$) in $\mathfrak{M} \times \mathfrak{V}$ is called a bisequence. It is

- Convergent if both ($\mathfrak{p}_n$) and ($\mathfrak{q}_n$) converge,
- Biconvergent if both converge to the same point.

(e) A bisequence ($\mathfrak{p}_n$, $\mathfrak{q}_n$) is Cauchy if:

$\forall \varepsilon > 0$, $\exists$ $n_0 \in N$, such that $\forall$ $n, m \ge n_0$, $w_\varphi(\mathfrak{p}_n, \mathfrak{q}_m) < \varepsilon$.

Definition 6. Let ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) and ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) be modular bipolar metric spaces.

- A map ξ : ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\Rightarrow$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) is left-continuous at $\mathfrak{p}_0 \in \mathfrak{M}_1$ if
  $\forall \varepsilon > 0, \exists \delta > 0$ such that $w_{\varphi_1}(\mathfrak{p}_0,\mathfrak{q}) < \delta \implies w_{\varphi_2}(\xi(\mathfrak{p}_0),\xi(\mathfrak{q})) < \varepsilon$, $\mathfrak{q} \in \mathfrak{V}_1$.
- A map ξ : ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\Rightarrow$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) is right-continuous at $\mathfrak{q}_0 \in \mathfrak{V}_1$ if
  $\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $w_{\varphi_1}(\mathfrak{p},\mathfrak{q}_0) < \delta \implies w_{\varphi_2}(\xi(\mathfrak{p}),\xi(\mathfrak{q}_0)) < \varepsilon$, $\mathfrak{p} \in \mathfrak{M}_1$.
- A map is continuous if it is both left- and right-continuous at every $\mathfrak{p} \in \mathfrak{M}_1$ and $\mathfrak{q} \in \mathfrak{V}_1$.
- A contravariant map ξ : ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\nearrow\searrow$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) is continuous iff it is continuous as a covariant map ξ : ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\rightrightarrows$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2})

Hence, a covariant or contravariant map ξ from ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) to ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) is continuous iff $\mathfrak{u}_n \to \mathfrak{v}$ in ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\Longrightarrow$ ξ ($\mathfrak{u}_n$) $\to$ ξ ($\mathfrak{v}$) in ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}).

Definition 7. Let ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) and ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) be modular bipolar metric spaces and $\theta > 0$.

A covariant map ξ: ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\rightrightarrows$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) such that $w_\varphi(\xi(\mathfrak{p}),\xi(\mathfrak{q})) \le \theta w_\varphi(\mathfrak{p},\mathfrak{q})$ for all $\mathfrak{p} \in \mathfrak{M}_1, \mathfrak{q} \in \mathfrak{V}_1$,

or a contravariant map ξ : ($\mathfrak{M}_1$, $\mathfrak{V}_1$, w_{φ_1}) $\nearrow\searrow$ ($\mathfrak{M}_2$, $\mathfrak{V}_2$, w_{φ_2}) such that

$w_\varphi(\xi(\mathfrak{q}),\xi(\mathfrak{p})) \le \theta w_\varphi(\mathfrak{p},\mathfrak{q})$ for all $p \in \mathfrak{M}_1, \mathfrak{q} \in \mathfrak{V}_1$ is called Lipschitz continuous.

If $\theta = 1$, then this covariant or contravariant map is said to be non-expansive, and if $\theta \in (0,1)$, it is called a contraction.

Definition 8. Let $(\mathfrak{M},\mathfrak{V},w_\varphi)$ be a modular bipolar metric space and $\mathfrak{T}$ be a self mapping. Then $\mathfrak{T}$ is said to be Chatterjea mapping if there exist $\alpha \in (0, \frac{1}{2})$ such that $w_\varphi(\xi\mathfrak{p},\xi\mathfrak{q}) \le \alpha[w_\varphi(\mathfrak{p},\xi\mathfrak{q}) + w_\varphi(\mathfrak{q},\xi\mathfrak{p})]$, for all $(\mathfrak{p},\mathfrak{q}) \in \mathfrak{M} \times \mathfrak{V}$.

Definition . Let $(\mathfrak{M},\mathfrak{V},w_\varphi)$ be a modular bipolar metric space, $\xi : (\mathfrak{M}^2,\mathfrak{V}^2) \rightrightarrows (\mathfrak{M},\mathfrak{V})$ be a covariant mapping. If $\xi(\mathfrak{p},\mathfrak{q}) = \mathfrak{p}$ and $\xi(\mathfrak{q},\mathfrak{p}) = \mathfrak{q}$ for $(\mathfrak{p},\mathfrak{q}) \in \mathfrak{M}^2 \cup \mathfrak{V}^2$ then $(\mathfrak{p},\mathfrak{q})$ is called a coupled fixed point of the mapping ξ.

Result and Discussion

Fixed point theorems

Theorem 3.1. Consider a complete modular bipolar metric space $(\mathfrak{M},\mathfrak{V},w_\varphi)$ and a covariant contraction $\xi : (\mathfrak{M},\mathfrak{V},w_\varphi) \rightrightarrows (\mathfrak{M},\mathfrak{V},w_\varphi)$ Thus, the function $\xi : \mathfrak{M} \cup \mathfrak{V} \to \mathfrak{M} \cup \mathfrak{V}$ has a distinctive fixed point.

Proof.

Assuming that ξ is a covariant contraction, there exists a $\theta \in (0,1)$ such that $w_\varphi(\xi(\mathfrak{p}),\xi(\mathfrak{q})) \le \theta w_\varphi(\mathfrak{p},q)$ for every $(\mathfrak{p},\mathfrak{q}) \in \mathfrak{M} \times \mathfrak{V}$.

Suppose $\mathfrak{p}_0 \in \mathfrak{M}$ and $\mathfrak{q}_0 \in \mathfrak{V}$. Assign $\xi(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$ and $\xi(\mathfrak{q}_n) = \mathfrak{q}_{n+1}$ to each $n \in N$. Now we need to demonstrate that $(\mathfrak{p}_n,\mathfrak{q}_n)$ is a bisequence on $(\mathfrak{M},\mathfrak{V},w_\varphi)$.

For every positive integer n and m, we have

$$\begin{aligned} w_\varphi(\mathfrak{p}_n,\mathfrak{q}_n) &= w_\varphi(\xi(\mathfrak{p}_{n-1}),\xi(\mathfrak{q}_{n-1})) \\ &\le \theta\cdot w_\varphi(\mathfrak{p}_{n-1},\mathfrak{q}_{n-1}) \\ &\le \theta^2\cdot w_\varphi(\mathfrak{p}_{n-2},\mathfrak{q}_{n-2}) \\ &\le \theta^n\cdot w_\varphi(\mathfrak{p}_0,\mathfrak{q}_0) \end{aligned}$$

$$\begin{aligned} w_\varphi(\mathfrak{p}_n,\mathfrak{q}_{n+1}) &= w_\varphi(\xi(\mathfrak{p}_{n-1}),\xi(\mathfrak{q}_n)) \\ &\le \theta\cdot w_\varphi(\mathfrak{p}_{n-1},\mathfrak{q}_n) \\ &\le \theta^2\cdot w_\varphi(\mathfrak{p}_{n-2},\mathfrak{q}_n) \\ &\le \theta^n\cdot w_\varphi(\mathfrak{p}_0,\mathfrak{q}_1) \end{aligned}$$

If $m>n$,

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_n) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + w_{\varphi_2}(\mathfrak{p}_n,\mathfrak{q}_{n+1}) + w_{\varphi_3}(\mathfrak{p}_n,\mathfrak{q}_n) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + \theta^n w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^n w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + \theta^n\left[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)\right] \end{aligned}$$

Meanwhile,

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_{n+1}) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + w_{\varphi_2}(\mathfrak{p}_{n+1},\mathfrak{q}_{n+2}) + w_{\varphi_3}(\mathfrak{p}_{n+1},\mathfrak{q}_{n+1}) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + \theta^{n+1} w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^{n+1} w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + \theta^{n+1}\left[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)\right] \end{aligned}$$

Likewise, we claim

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_{m-1}) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + w_{\varphi_2}(\mathfrak{p}_{m-1},\mathfrak{q}_m) + w_{\varphi_3}(\mathfrak{p}_{m-1},\mathfrak{q}_{m-1}) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + \theta^{m-1} w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^{m-1} w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + \theta^{m-1}\left[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)\right] \end{aligned}$$

Determine $D=w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1)+w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)$ using the equations above, and Using the fact that $\varphi_1=\varphi_2=\varphi_3=\cdots=\varphi_{m-n}=\frac{\varphi}{m-n}$, we obtain

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_n) &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + \theta^n D \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + \theta^n D + \theta^{n+1} D \\ &\le w_{\frac{\varphi}{m-n}}(\mathfrak{p}_m,\mathfrak{q}_m) + \theta^n D + \theta^{n+1} D + \cdots + \theta^{m-1} D \\ &\le \theta^n D + \theta^{n+1} D + \cdots + \theta^{m-1} D + \theta^m D \\ &\le \left[\frac{\theta^n D}{1-\theta}\right] \to 0 \end{aligned}$$

Consequently, $(\mathfrak{p}_n,\mathfrak{q}_n)$ is a Cauchy bisequence.

Since $(\mathfrak{M},\mathfrak{V},w_\varphi)$ is complete, the sequence $(\mathfrak{p}_n,\mathfrak{q}_n)$ converges and biconverges to a point. It is ensured that $\xi(\mathfrak{q}_n)$ has a distinct limit if $\tau \in\mathfrak{M}\cap\mathfrak{V}$ and $\xi(\mathfrak{q}_n)=(\mathfrak{q}_{n+1})\to\tau\in\mathfrak{M}\cap\mathfrak{V}$, with $\xi(\mathfrak{q}_n)\to\xi(\tau)$ because ξ is continuous. Therefore, $\xi(\tau)=\tau$; it follows that τ is a fixed point of ξ.

If $g(\nu)=\nu\Rightarrow\nu\in\mathfrak{M}\cap\mathfrak{V}$ and we have the inequality $w_\varphi(\tau,\nu)= w_\varphi(\xi(\tau),\xi(\nu))\le\theta\, w_\varphi(\tau,\nu)$, where $0 < \theta < 1$, if ν is any fixed point of ξ.

Consequently, we conclude that $w_\varphi(\tau,\nu) = 0$.

Hence, $\tau=\nu$.

Theorem 3.2. Consider a complete modular bipolar metric space $(\mathfrak{M},\mathfrak{V},w_\varphi)$ and a contravariant contraction $\xi: (\mathfrak{M},\mathfrak{V},w_\varphi) \nearrow\searrow (\mathfrak{M},\mathfrak{V},w_\varphi)$. Thus, the function $\xi: \mathfrak{M}\cup\mathfrak{V}\to\mathfrak{M}\cup\mathfrak{V}$ has a distinctive fixed point.

Proof.

Assuming that ξ is a contravariant contraction, there exists a $\theta \in (0,1)$ such that

$w_\varphi(\xi(\mathfrak{q}),\xi(\mathfrak{p}))\le\theta w_\varphi(\mathfrak{p},\mathfrak{q})$ for every $(\mathfrak{p},\mathfrak{q})\in\mathfrak{M}\times\mathfrak{V}$.

Suppose $\mathfrak{p}_0\in\mathfrak{M}$. Assign $\xi(\mathfrak{p}_n) = \mathfrak{q}_n$ and $\xi(\mathfrak{q}_n) = \mathfrak{p}_{n+1}$ to each $n\in N$. Now, we need to demonstrate that $(\mathfrak{p}_n, \mathfrak{q}_n)$ is a bisequence on $(\mathfrak{M},\mathfrak{V},w_\varphi)$.

For every positive integer n and m, we have

$$\begin{aligned} w_\varphi(\mathfrak{p}_n,\mathfrak{q}_n) &= w_\varphi(\xi(\mathfrak{q}_{n-1}),\xi(\mathfrak{p}_n)) \\ &\le \theta\cdot w_\varphi(\mathfrak{p}_n,\mathfrak{q}_{n-1}) \\ &\le \theta^2\cdot w_\varphi(\mathfrak{p}_{n-1},\mathfrak{q}_{n-1}) \\ &\le \theta^{2n}\cdot w_\varphi(\mathfrak{p}_0,\mathfrak{q}_0) \end{aligned}$$

$$\begin{aligned} w_\varphi(\mathfrak{p}_n,\mathfrak{q}_{n+1}) &= w_\varphi(\xi(\mathfrak{p}_{n-1}),\xi(\mathfrak{q}_n)) \\ &\le \theta\cdot w_\varphi(\mathfrak{p}_{n-1},\mathfrak{q}_n) \\ &\le \theta^{2n}\cdot w_\varphi(\mathfrak{p}_0,\mathfrak{q}_1) \end{aligned}$$

If $m>n$,

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_n) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + w_{\varphi_2}(\mathfrak{p}_n,\mathfrak{q}_{n+1}) + w_{\varphi_3}(\mathfrak{p}_n,\mathfrak{q}_n) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + \theta^{2n} w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^{2n} w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+1}) + \theta^{2n}[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)] \end{aligned}$$

Meanwhile,

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_{n+1}) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + w_{\varphi_2}(\mathfrak{p}_{n+1},\mathfrak{q}_{n+2}) + w_{\varphi_3}(pn+1,\mathfrak{q}_{n+1}) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + \theta^{2n+2} w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^{2n+2} w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_{n+2}) + \theta^{2n+2}\left[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)\right] \end{aligned}$$

Likewise, we claim

$$\begin{aligned} w_\varphi(\mathfrak{p}_m,\mathfrak{q}_{m-1}) &= w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + w_{\varphi_2}(\mathfrak{p}_{m-1},\mathfrak{q}_m) + w_{\varphi_3}(p_{m-1},\mathfrak{q}_{m-1}) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + \theta^{2m-2} w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + \theta^{2m-2} w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0) \\ &\le w_{\varphi_1}(\mathfrak{p}_m,\mathfrak{q}_m) + \theta^{2m-2}\left[w_{\varphi_2}(\mathfrak{p}_0,\mathfrak{q}_1) + w_{\varphi_3}(\mathfrak{p}_0,\mathfrak{q}_0)\right] \end{aligned}$$

Determine $D= w_{\varphi_2}\ (\mathfrak{p}_0,\ \mathfrak{q}_1)\ +\ w_{\varphi_3}\ (\mathfrak{p}_0,\ \mathfrak{q}_0)$ using the equations above, and
Using the fact that $\varphi_1=\varphi_2=\varphi_3=\cdots=\varphi_{m-n}=\dfrac{\varphi}{m-n}$, we obtain

$$
\begin{aligned}
w_{\varphi}\left(\mathfrak{p}_m,\mathfrak{q}_n\right) &\le w_{\varphi_1}\left(\mathfrak{p}_m,\mathfrak{q}_{n+1}\right)+\theta^{2n}\ D\\
&\le w_{\varphi_1}\left(\mathfrak{p}_m,\mathfrak{q}_{n+2}\right)+\theta^{2n}D+\theta^{2n+2}D\\
&\le w_{\frac{\varphi}{m-n}}\left(\mathfrak{p}_m,\mathfrak{q}_m\right)+\theta^{2n}D+\theta^{2n+2}D+\cdots+\theta^{2m-2}D\\
&\le \theta^{2n}D+\theta^{2n+2}D+\cdots+\theta^{2m-2}D+\theta^{2m}D\\
&\le\left[\frac{\theta^{2n}D}{1-\theta}\right]\to 0
\end{aligned}
$$

Consequently, the sequence $(\mathfrak{p}_n,\ \mathfrak{q}_n)$ is a Cauchy bisequence.

Since $(\mathfrak{M},\mathfrak{V},w_{\varphi})$ is complete modular bipolar metric space, the sequence $(\mathfrak{p}_n,\mathfrak{q}_n)$ converges and biconverges to a point. It is ensured that $(\mathfrak{p}_n)$ and $(\mathfrak{q}_n)$ has a distinct limit if $(\mathfrak{p}_n)\to\tau$, $(\mathfrak{q}_n)\to\tau$, where $\tau\in\mathfrak{M}\cap\mathfrak{V}$, with $(\mathfrak{p}_n)\to\tau$ implies that $(\mathfrak{q}_n)=\xi((\mathfrak{p}_n))\to\xi(\tau)\Rightarrow(\mathfrak{q}_n)\to\tau$ because ξ is continuous. Therefore, $\xi(\tau)=\tau$.

It follows that τ is a fixed point of ξ.

If ν is another fixed point of ξ , then $\xi(\nu)=\nu\Rightarrow\nu\in\mathfrak{M}\cap\mathfrak{V}$, such that

$$
\begin{aligned}
w_{\varphi}(\tau,\nu)&=w_{\varphi}\left(\xi(\tau),\ \xi(\nu)\right)\\
&\le\theta\cdot w_{\varphi}(\tau,\nu).
\end{aligned}
$$

Since $0<\theta<1$, it follows that $w_{\varphi}(\tau,\nu)=0$, and therefore,

$$\tau=\nu.$$

Example: Let $\mathfrak{M}=\{3,5,8,12\}$ and $\mathfrak{V}=\{4,7,10,12\}$ equipped with $w_{\varphi}(\mathfrak{p},\mathfrak{q})=|\mathfrak{p}-\mathfrak{q}|$. Then $(\mathfrak{M},\mathfrak{V},w_{\varphi})$ is a modular bipolar metric space.

The contravariant mapping $\xi:\mathfrak{M}\cup\mathfrak{V}\to\mathfrak{M}\cup\mathfrak{V}$ is defined by

$$\xi(z)=\begin{cases}12\ , z\in M\cup\{10\}\\ 8, otherwise\end{cases}$$

Thus, it satisfies the inequality $w_{\varphi}\ (\xi(\mathfrak{q}),\xi(\mathfrak{p}))\le\theta\cdot w_{\varphi}\ (\mathfrak{p},\mathfrak{q})$, for some

$\theta\in(0,1)$. By Theorem 3.2, ξ has a unique fixed point.

Using a covariant contraction, we provide a Chatterjea-type fixed point result, extending classical results to the realm of modular bipolar metric spaces.

Theorem 3.3. Consider a complete modular bipolar metric space $(\mathfrak{M},\mathfrak{V},w_{\varphi})$, and a covariant contraction $\xi:(\mathfrak{M},\mathfrak{V},w_{\varphi})\rightrightarrows(\mathfrak{M},\mathfrak{V},w_{\varphi})$ satisfying $w_{\varphi}(\xi\mathfrak{p},\xi\mathfrak{q})\le\alpha\left[w_{\varphi}(\mathfrak{p},\xi\mathfrak{q})+w_{\varphi}(\mathfrak{q},\xi\mathfrak{p})\right]$, for every $(\mathfrak{p},\mathfrak{q})\in\mathfrak{M}\times\mathfrak{V}$, where $\alpha\in(0,\frac{1}{2})$. Then, the mapping $\xi:\mathfrak{M}\cup\mathfrak{V}\to\mathfrak{M}\cup\mathfrak{V}$ has a distinct fixed point.

Proof.

Assume $\mathfrak{p}_0\in\mathfrak{M}$ and define sequences $\{\mathfrak{p}_n\}$ and $\{\mathfrak{q}_n\}$ recursively by
$\mathfrak{p}_{n+1}=\xi(\mathfrak{p}_n)$ and $\mathfrak{q}_{n+1}=\xi(\mathfrak{q}_n)$, for all $n\ge 0$.

$$
\begin{aligned}
w_{\varphi}\ (\mathfrak{p}_n,\mathfrak{q}_n)\ &=\ w_{\varphi}\ \left(\xi(\mathfrak{p}_{n-1}),\xi(\mathfrak{q}_{n-1})\right)\\
&\le\alpha\left[w_{\varphi}\left(\mathfrak{p}_{n-1},\xi(\mathfrak{q}_{n-1})\right)+w_{\varphi}\left(\mathfrak{q}_{n-1},\xi(\mathfrak{p}_{n-1})\right)\right]\\
&=\alpha\left[w_{\varphi}\left(\mathfrak{p}_{n-1},\mathfrak{q}_n\right)+w_{\varphi}\left(\mathfrak{q}_{n-1},\mathfrak{p}_n\right)\right]\\
&\le\alpha\cdot w_{\varphi}\left(\mathfrak{p}_{n-1},\mathfrak{q}_{n-1}\right)\\
&\le\alpha^n\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_0\right).
\end{aligned}
$$

$$
\begin{aligned}
w_{\varphi}\ \left(\mathfrak{p}_{n-1},\mathfrak{q}_n\right)&=w_{\varphi}\left(\xi(\mathfrak{p}_{n-2}),\xi(\mathfrak{q}_{n-1})\right)\\
&\le\alpha\left[w_{\varphi}\left(\mathfrak{p}_{n-2},\xi(\mathfrak{q}_{n-1})\right)+w_{\varphi}\left(\mathfrak{q}_{n-1},\xi(\mathfrak{p}_{n-2})\right)\right]\\
&=\alpha\left[w_{\varphi}\left(\mathfrak{p}_{n-2},\mathfrak{q}_n\right)+w_{\varphi}\left(\mathfrak{q}_{n-1},\mathfrak{p}_{n-1}\right)\right]\\
&\le\alpha\cdot w_{\varphi}\left(\mathfrak{p}_{n-2},\mathfrak{q}_{n-1}\right)\\
&\le\alpha^{n-1}\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_1\right).
\end{aligned}
$$

For every positive integers m and n , we consider two cases:

Case 1: $m>n$

$$
\begin{aligned}
w_{\varphi}\left(\mathfrak{p}_n,\mathfrak{q}_m\right)&=w_{\varphi_1}\left(\mathfrak{p}_n,\mathfrak{q}_n\right)+w_{\varphi_2}\left(\mathfrak{p}_{n+1},\mathfrak{q}_n\right)+w_{\varphi_3}\left(\mathfrak{p}_{n+1},\mathfrak{q}_m\right)\\
&\le\left(\alpha^n+\alpha^{n+1}\right)\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_0\right)+w_{\varphi_3}\left(\mathfrak{p}_{n+1},\mathfrak{q}_m\right)\\
&\le\left(\alpha^n+\alpha^{n+1}+\cdots+\alpha^m\right)\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_0\right).
\end{aligned}
$$

Case 2: $m<n$

$$
\begin{aligned}
w_{\varphi}\left(\mathfrak{p}_n,\mathfrak{q}_m\right)&=w_{\varphi_1}\left(p_{m+1},\mathfrak{q}_m\right)+w_{\varphi_2}\left(p_{m+1},\mathfrak{q}_{m+1}\right)+w_{\varphi_3}\left(\mathfrak{p}_n,\mathfrak{q}_{m+1}\right)\\
&\le\left(\alpha^{2m+1}+\alpha^{2m+2}\right)\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_0\right)+w_{\varphi_3}\left(\mathfrak{p}_n,\mathfrak{q}_{m+1}\right)\\
&\le\left(\alpha^{m+1}+\alpha^{m+2}+\cdots+\alpha^{n+1}\right)\cdot w_{\varphi}\left(\mathfrak{p}_0,\mathfrak{q}_0\right).
\end{aligned}
$$

Since $\alpha\in(0,1)$, the sequence $w_{\varphi}(\mathfrak{p}_n,\mathfrak{q}_m)$ can be made arbitrarily small by choosing sufficiently large values for m and n . Therefore, $(\mathfrak{p}_n,\mathfrak{q}_m)$ is a Cauchy bisequence.

Given that $(\mathfrak{M},\mathfrak{V},w_{\varphi})$ is complete, $(\mathfrak{p}_n,\mathfrak{q}_m)$ converges. This implies that it also biconverges, as it is a convergent Cauchy bisequence.

Let u be the limit point of the sequence $(\mathfrak{p}_n,\mathfrak{q}_m)$. Then we have

$\xi(\mathfrak{q}_n)=\mathfrak{q}_{n+1}\to\mathfrak{u}$, where $\mathfrak{u}\in\mathfrak{M}\cap\mathfrak{V}$.

Since ξ is continuous, it follows that $\xi(\mathfrak{q}_n)\to\xi(\mathfrak{u})$, which implies $\xi(\mathfrak{u})=\mathfrak{u}$, so ξ has a fixed point.

If $\mathfrak{v}$ is any fixed point of ξ, then $\xi(\mathfrak{v})=\mathfrak{v}$ implies $\mathfrak{v}\in\mathfrak{M}\cap\mathfrak{V}$,

$$w_\varphi(\xi\mathfrak{u},\xi\mathfrak{v})\le\alpha\left[w_\varphi(\mathfrak{u},\xi\mathfrak{v})+w_\varphi(\mathfrak{v},\xi\mathfrak{u})\right],$$

$$w_\varphi(\mathfrak{u},\mathfrak{v})\le\alpha\left[w_\varphi(\mathfrak{u},\mathfrak{v})+w_\varphi(\mathfrak{v},\mathfrak{u})\right],$$

$$w_\varphi(\mathfrak{u},\mathfrak{v})\le 2\alpha w_\varphi(\mathfrak{u},\mathfrak{v})$$

$$(1-2\alpha)w_\varphi(\mathfrak{u},\mathfrak{v})\le 0$$

Since, $0<\alpha<\frac{1}{2}$, it follows that

$$w_\varphi(\mathfrak{u},\mathfrak{v})=0,$$

$$\mathfrak{u}=\mathfrak{v}.$$

Corollary 3.4. Consider a complete modular bipolar metric space $(\mathfrak{M},\mathfrak{V},w_\varphi)$ and a contravariant contraction $\xi:(\mathfrak{M},\mathfrak{V},w_\varphi)\nearrow\searrow(\mathfrak{M},\mathfrak{V},w_\varphi)$ satisfying

$w_\varphi(\xi\mathfrak{q},\xi\mathfrak{p})\le\alpha\left[w_\varphi(\mathfrak{p},\xi\mathfrak{p})+w_\varphi(\mathfrak{q},\xi\mathfrak{q})\right]$, for every $(\mathfrak{p},\mathfrak{q})\in\mathfrak{M}\times B$,

where $\alpha\in(0,\frac{1}{2})$. Then, the mapping ξ: $\mathfrak{M}\cup\mathfrak{V}\to\mathfrak{M}\cup\mathfrak{V}$ has a distinct fixed point.

Example

Let $\xi:(\mathfrak{M},\mathfrak{V},w_\varphi)\rightrightarrows(\mathfrak{M},\mathfrak{V},w_\varphi)$ where $(\mathfrak{M},\mathfrak{V},w_\varphi)$ be a complete modular bipolar metric space. Let $\mathfrak{M}$ be the set of all real numbers of the form $\frac{n}{2}$ and $\mathfrak{V}$ be the set of all finite subsets of rational numbers in [0, 5]. Define $w_\varphi(\mathfrak{x},A)=|\mathfrak{x}-min(A)|+|\mathfrak{x}-max(A)|$, and ξ (A) = $\{\frac{\min(A)+\max(A)+1}{3}\}$.
The mapping ξ satisfies the inequality

$w_\varphi(\xi\mathfrak{x},\xi\mathfrak{y})\le\alpha\left[w_\varphi(\mathfrak{x},\xi\mathfrak{y})+w_\varphi(\mathfrak{y},\xi\mathfrak{x})\right]$, for all $(\mathfrak{x},\mathfrak{y})\in\mathfrak{M}\times\mathfrak{V}$.
Then ξ has a unique fixed point.
Solution: Let $\mathfrak{M}=\left\{\ldots,\frac{-3}{2},-1,\frac{-1}{2},0,\frac{1}{2},1,\frac{3}{2}\ldots\right\},\mathfrak{V}=\{A\subset Q\cap[0,5]\}$.
Given $w_\varphi(\mathfrak{x},A)=|\mathfrak{x}-min(A)|+|\mathfrak{x}-max(A)|$,

$$w_\varphi(\xi(A),\xi(B))=\left|\frac{\min(A)+\max(A)+1}{3}-\frac{\min(B)+\max(B)+1}{3}\right| \quad (1)$$

and

$$w_\varphi(A,B)=|min(A)-min(B)|+|max(A)-max(B)| \quad (2)$$

From (1) and (2), it follows that

$$w_\varphi(\xi(A),\xi(B))\le\frac{1}{3}\ w_\varphi(A,B)$$

We can rewrite the inequality as

$w_\varphi(\xi(A),\xi(B))\le\frac{1}{3}\left[w_\varphi(\mathfrak{x},\xi y)+w_\varphi(y,\xi\mathfrak{x})\right]$. Hence, $\alpha=\frac{1}{3}$.

For uniqueness, suppose $\xi(A^*)=A^*$, then

$$a=\frac{\min(A^*)+\max(A^*)+1}{3}\quad a=\frac{a+a+1}{3}$$

$$a=1.$$

Thus, ξ has the unique fixed point 1.

Coupled fixed point theorem

Theorem 3.5. Consider $(\mathfrak{M},\mathfrak{V},w_\varphi)$ be a complete modular bipolar metric space, the mapping $\xi:(\mathfrak{M}^2,\mathfrak{V}^2)\rightrightarrows(\mathfrak{M},\mathfrak{V})$ be a covariant contraction and $\mathfrak{a}+\mathfrak{b}<1$ where $\mathfrak{a}$ and $\mathfrak{b}$ are non-negative constants.
If ξ satisfies the condition

$$w_\varphi(\xi(\mathfrak{x},\mathfrak{y}),\xi(\mathfrak{u},\mathfrak{v}))\le\mathfrak{a}\,w_\varphi(\mathfrak{x},\mathfrak{u})+\mathfrak{b}\,w_\varphi(\mathfrak{y},\mathfrak{v}) \quad (1)$$

where $\mathfrak{x},\mathfrak{y}\in\mathfrak{M}$, $\mathfrak{u},\mathfrak{v}\in\mathfrak{V}$, then ξ has a unique coupled fixed point.

Proof.

Assume $\mathfrak{x}_0,\mathfrak{y}_0\in\mathfrak{M}$ and $\mathfrak{u}_0$, $\mathfrak{v}_0\in\mathfrak{V}$. We choose $\mathfrak{x}_1,\mathfrak{y}_1\in\mathfrak{M}$ with $\mathfrak{x}_1=\xi(\mathfrak{x}_0,\mathfrak{y}_0)$ and $\mathfrak{y}_1=\xi(\mathfrak{y}_0,\mathfrak{x}_0)$. Subsequently, assuming that $\mathfrak{u}_1,\mathfrak{v}_1\in\mathfrak{V}$ with $\mathfrak{u}_1=\xi(\mathfrak{u}_0,\mathfrak{v}_0)$ and $\mathfrak{v}_1=\xi(\mathfrak{v}_0,\mathfrak{u}_0)$ and $\mathfrak{x}_2,\mathfrak{y}_2\in\mathfrak{M}$ and $\mathfrak{u}_2,\mathfrak{v}_2\in\mathfrak{V}$ with $\mathfrak{x}_2=\xi(\mathfrak{x}_1,\mathfrak{y}_1)$,

$\mathfrak{y}_2=\xi(\mathfrak{y}_1,\mathfrak{x}_1)$, $\mathfrak{u}_2=\xi(\mathfrak{u}_1,\mathfrak{v}_1)$, $\mathfrak{v}_2=\xi(\mathfrak{v}_1,\mathfrak{u}_1)$. Following this procedure, we get the bisequences $(\mathfrak{x}_n,\mathfrak{y}_n)$ and $(\mathfrak{u}_n,\mathfrak{v}_n)$ with

$$\mathfrak{x}_{n+1}=\xi(\mathfrak{x}_n,\mathfrak{y}_n);\mathfrak{y}_{n+1}=\xi(\mathfrak{y}_n,\mathfrak{x}_n);$$

$$\mathfrak{u}_{n+1}=\xi(\mathfrak{u}_n,\mathfrak{v}_n);\mathfrak{v}_{n+1}=\xi(\mathfrak{v}_n,\mathfrak{u}_n)$$

for all $n\in N^+$. Let $\mathfrak{a}+\mathfrak{b}=\Gamma$. Then applying (1), we obtain

$$w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n+1})=w_\varphi(\xi(\mathfrak{x}_{n-1},\mathfrak{y}_{n-1}),\xi(\mathfrak{u}_n,\mathfrak{v}_n))$$

$$\le\mathfrak{a}\,w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_n)+\mathfrak{b}\,w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_n) \quad (2)$$

and

$$w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n+1})=w_\varphi(\xi(\mathfrak{y}_{n-1},\mathfrak{x}_{n-1}),\xi(\mathfrak{v}_n,\mathfrak{u}_n))$$

$$\le\mathfrak{a}\,w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_n)+\mathfrak{b}\,w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_n) \quad (3)$$

Let $\acute{o}_n=w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n+1})$
By combining equations (2) and (3) we obtain,

$$\sigma_n=w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n+1})\le\mathfrak{a}\,w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_n)$$

$$+\mathfrak{b}\,w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_n))+\mathfrak{a}\,w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_n)+\mathfrak{b}\,w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_n)$$

$$=(\mathfrak{a}+\mathfrak{b})(w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_n))$$

$$=\Gamma\sigma_{n-1}$$

Next, we get that

$$0\le\sigma_n\le\Gamma\sigma_{n-1}\le\Gamma^2\sigma_{n-2}\le\cdots\le\Gamma^n\sigma_0 \tag{4}$$

Alternatively,

$$w_\varphi(\mathfrak{x}_{n+1},\mathfrak{u}_n)=w_\varphi(\xi(\mathfrak{x}_n,\ \mathfrak{y}_n),\xi(\mathfrak{u}_{n-1},\mathfrak{v}_{n-1}))$$
$$\le\mathfrak{a}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n-1}) \tag{5}$$

$$\begin{aligned}w_\varphi(\mathfrak{y}_{n+1},\mathfrak{v}_n)&=w_\varphi(\xi(\mathfrak{y}_n,\mathfrak{x}_n),\xi(\mathfrak{v}_{n-1},\mathfrak{u}_{n-1}))\\&\le\mathfrak{a}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n-1})\end{aligned} \tag{6}$$

for every $n\in N^+$.

Define

$\alpha_n=w_\varphi(\mathfrak{x}_{n+1},\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_{n+1},\mathfrak{v}_n)$

By combining equations (5) and (6) we obtain,

$$\begin{aligned}\alpha_n&=w_\varphi(\mathfrak{x}_{n+1},\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_{n+1},\mathfrak{v}_n)\\&\le\mathfrak{a}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n-1})+\mathfrak{a}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n-1})\\&=(\mathfrak{a}+\mathfrak{b})(w_\varphi(\mathfrak{x}_n,\mathfrak{u}_{n-1})+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_{n-1}))\\&=\Gamma\alpha_{n-1}\end{aligned}$$

It follows that $0\le\alpha_n\le\Gamma\alpha_{n-1}\le\Gamma^2\alpha_{n-2}\le\cdots\le\Gamma^n\alpha_0$. (7)

Also,

$$\begin{aligned}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_n)&=w_\varphi(\xi(\mathfrak{x}_{n-1},\mathfrak{y}_{n-1}),\xi(\mathfrak{u}_{n-1},\mathfrak{v}_{n-1}))\\&\le\mathfrak{a}w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_{n-1})\end{aligned} \tag{8}$$

$$\begin{aligned}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_n)&=w_\varphi(\xi(\mathfrak{y}_{n-1},\mathfrak{x}_{n-1}),\xi(\mathfrak{v}_{n-1},\mathfrak{u}_{n-1}))\\&\le\mathfrak{a}w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_{n-1})\end{aligned} \tag{9}$$

for every $n\in N^+$.

Let $\beta_n=w_\varphi(\mathfrak{x}_n,\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_n)$

By combining equations (8) and (9) we obtain the following,

$$\begin{aligned}\beta_n&=w_\varphi(\mathfrak{x}_n,\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_n)\\&\le\mathfrak{a}w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_{n-1})+\mathfrak{a}w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_{n-1})+\mathfrak{b}w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_{n-1})\\&=(a+b)(w_\varphi(\mathfrak{x}_{n-1},\mathfrak{u}_{n-1})+w_\varphi(\mathfrak{y}_{n-1},\mathfrak{v}_{n-1}))\\&=\Gamma\beta_{n-1}\end{aligned}$$

It follows that

$$0\le\beta_n\le\Gamma\beta_{n-1}\le\Gamma^2\beta_{n-2}\le\cdots\le\Gamma^n\beta_0. \tag{10}$$

Assume that $\sigma_0,\beta_0,\alpha_0>0$. Let $m,n\in N$ with $n<m$. Then there exists $n_{\frac{\varphi}{m-n}}\in N^+$ such that

ó $w_{\frac{\varphi}{m-n}}(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_n,\mathfrak{v}_{n+1})= {}_{n}$

$$\begin{aligned}&w_{\frac{\varphi}{m-n}}(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_n,\mathfrak{v}_{n+1})=\sigma_n\\&w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_n)+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{n+1},\mathfrak{v}_n)=\alpha_n\\&w_{\frac{\varphi}{m-n}}(\mathfrak{x}_n,\mathfrak{u}_n)+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_n,\mathfrak{v}_n)=\beta_n\end{aligned} \tag{11}$$

for every $\frac{\varphi}{m-n}>0$ *and* $n\ge n_{\frac{\varphi}{m-n}}$.

Invoking property (3c) of the modular bipolar metric space, we obtain

$$w_\varphi(\mathfrak{x}_n,\mathfrak{u}_m)\le w_{\frac{\varphi}{m-n}}(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})+\cdots+w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{m-1},\mathfrak{u}_m)$$

$$w_\varphi(\mathfrak{y}_n,\mathfrak{v}_m)\le w_{\frac{\varphi}{m-n}}(\mathfrak{y}_n,\mathfrak{v}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{n+1},\mathfrak{v}_{n+1})+\cdots+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{m-1},\mathfrak{v}_m)$$

$$w_\varphi(\mathfrak{x}_m,\mathfrak{u}_n)\le w_{\frac{\varphi}{m-n}}(\mathfrak{x}_m,\mathfrak{u}_{m-1})+w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{m-1},\mathfrak{u}_{m-1})+\cdots+w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_n)$$

$$w_\varphi(\mathfrak{y}_m,\mathfrak{v}_n)\le w_{\frac{\varphi}{m-n}}(\mathfrak{y}_m,\mathfrak{v}_{m-1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{m-1},\mathfrak{v}_{m-1})+\cdots+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{n+1},\mathfrak{v}_n) \tag{12}$$

for every $n,m\in N$ and $n<m$. Hence, combining (4),(7),(10),(11) and (12), we deduce

$$\begin{aligned}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_m)+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_m)\le(&w_{\frac{\varphi}{m-n}}(\mathfrak{x}_n,\mathfrak{u}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_n,\mathfrak{v}_{n+1}))+\\&\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{n+1},\mathfrak{v}_{n+1})\right)+\ldots\\&\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{m-1},\mathfrak{u}_{m-1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{m-1},\mathfrak{v}_{m-1})\right)+\\&\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{m-1},\mathfrak{u}_m)+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{m-1},\mathfrak{v}_m)\right))\end{aligned}$$

$$=\sigma_n+\beta_{n+1}+\sigma_{n+1}+\cdots+\beta_{m-1}+\sigma_{m-1}$$
$$\leq\Gamma^n\sigma_0+\Gamma^{n+1}\beta_0+\Gamma^{n+1}\sigma_0+\cdots+\Gamma^{m-1}\beta_0+\Gamma^{m-1}\sigma_0$$
$$=\left(\Gamma^n+\Gamma^{n+1}+\cdots+\Gamma^{m-1}\right)\sigma_0+\left(\Gamma^{n+1}+\Gamma^{n+2}+\cdots+\Gamma^{m-1}\right)\beta_0 \quad (13)$$
$$\leq\frac{\Gamma^n}{1-\Gamma}\sigma_0+\frac{\Gamma^{n+1}}{1-\Gamma}\beta_0$$

and

$$w_\varphi(\mathfrak{x}_m,\mathfrak{u}_n)+w_\varphi(\mathfrak{y}_m,\mathfrak{v}_n)\leq\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_m,\mathfrak{u}_{m-1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_m,\mathfrak{v}_{m-1})\right)+$$
$$\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{m-1},\mathfrak{u}_{m-1})+w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{m-1},\mathfrak{v}_{m-1})\right)+\ldots$$
$$\left(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})w_{\frac{\varphi}{m-n}}(\mathfrak{y}_{n+1},\mathfrak{v}_{n+1})\right)+$$
$$(w_{\frac{\varphi}{m-n}}(\mathfrak{x}_{n+1},\mathfrak{u}_n)+w_{\frac{\varphi}{m-n}}((\mathfrak{y}_{n+1},\mathfrak{v}_n))$$
$$=\alpha_{m-1}+\beta_{m-1}+\cdots+\alpha_{n+1}+\beta_{n+1}+\alpha_n$$
$$\leq\Gamma^{m-1}\alpha_0+\Gamma^{m-1}\beta_0+\cdots+\Gamma^{n+1}\alpha_0+\Gamma^{n+1}\beta_0+\Gamma^n\alpha_0 \quad (14)$$
$$=\left(\Gamma^n+\Gamma^{n+1}+\cdots+\Gamma^{m-1}\right)\alpha_0+\left(\Gamma^{n+1}+\Gamma^{n+2}+\cdots+\Gamma^{m-1}\right)\beta_0$$
$$\leq\frac{\Gamma^n}{1-\Gamma}\alpha_0+\frac{\Gamma^{n+1}}{1-\Gamma}\beta_0$$

for $n<m$. Since there exists n_0 such that $\frac{\Gamma^n}{1-\Gamma}\sigma_0+\frac{\Gamma^{n+1}}{1-\Gamma}\beta_0<\frac{\varepsilon}{3}$ and

$$\frac{\Gamma^n}{1-\Gamma}\alpha_0+\frac{\Gamma^{n+1}}{1-\Gamma}\beta_0<\frac{\varepsilon}{3}.$$

Combining (13) and (14) we obtain

$$w_\varphi(\mathfrak{x}_n,\mathfrak{u}_m)+w_\varphi(\mathfrak{y}_n,\mathfrak{v}_m)<\frac{\varepsilon}{3}$$

for an arbitrary $\varepsilon>0$ and for all $n,m\geq n_0$. Thus $(\mathfrak{x}_n,\mathfrak{u}_n)$ and $(\mathfrak{y}_n,\mathfrak{v}_n)$ are Cauchy bisequences. By the completeness of $(\mathfrak{M},\mathfrak{V},w_\varphi)$, there exist elements $x,y\in\mathfrak{M}$, $u,v\in\mathfrak{V}$ such that

$$\lim_{n\to\infty}\mathfrak{x}_n=\mathfrak{u},$$
$$\lim_{n\to\infty}\mathfrak{y}_n=\mathfrak{v},$$
$$\lim_{n\to\infty}\mathfrak{u}_n=\mathfrak{x},$$
$$\lim_{n\to\infty}\mathfrak{v}_n=\mathfrak{y}. \quad (15)$$

There exists $n_1\in N$ with $w_{\frac{\varphi}{3}}(\mathfrak{x}_n,\mathfrak{u})<\frac{\varepsilon}{3}, w_{\frac{\varphi}{3}}(\mathfrak{y}_n,\mathfrak{v})<\frac{\varepsilon}{3}$, $w_{\frac{\varphi}{3}}(\mathfrak{x},\mathfrak{u}_n)<\frac{\varepsilon}{3}$, $w_{\frac{\varphi}{3}}(\mathfrak{y},\mathfrak{v}_n)<\frac{\varepsilon}{3}$, for all $n\geq n_1$ and every $\varepsilon>0$. Since $(\mathfrak{x}_n,\mathfrak{u}_n)$ and $(\mathfrak{y}_n,\mathfrak{v}_n)$ are Cauchy bisequences, it follows that $w_{\frac{\varphi}{3}}(\mathfrak{x}_n,\mathfrak{u}_n)<\frac{\varepsilon}{3}, w_{\frac{\varphi}{3}}(\mathfrak{y}_n,\mathfrak{v}_n)<\frac{\varepsilon}{3}$.

Moreover, from (1), we obtain

$$w_\varphi(\xi(\mathfrak{x},\mathfrak{y}),\mathfrak{u})\leq w_{\frac{\varphi}{3}}(\xi(\mathfrak{x},\mathfrak{y}),\mathfrak{u}_{n+1})+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u})$$
$$=w_{\frac{\varphi}{3}}(\xi(\mathfrak{x},\mathfrak{y}),\xi(\mathfrak{u}_n,\mathfrak{v}_n))+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u})$$
$$\leq\mathfrak{a}\,w_{\frac{\varphi}{3}}(\mathfrak{x},\mathfrak{u}_n)+\mathfrak{b}\,w_{\frac{\varphi}{3}}(\mathfrak{y},\mathfrak{v}_n)+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u}_{n+1})+w_{\frac{\varphi}{3}}(\mathfrak{x}_{n+1},\mathfrak{u})$$
$$<\mathfrak{a}.\frac{\varepsilon}{3}+\mathfrak{b}.\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$
$$=\Gamma\cdot\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}<\varepsilon$$

for each $n\in N$ with $\Gamma<1$, we have $w_\varphi(\xi(\mathfrak{x},\mathfrak{y}),\mathfrak{u})$ = 0.

Therefore, $\xi(\mathfrak{x},\mathfrak{y})=\mathfrak{u}$. Likewise, it follows that $\xi(\mathfrak{y},\mathfrak{x})=\mathfrak{v}$, $\xi(\mathfrak{u},\mathfrak{v})=\mathfrak{x}$ and $\xi(v,\mathfrak{u})=\mathfrak{y}$. Alternatively, from (15) we obtain

$$w_\varphi(\mathfrak{x},\mathfrak{u})=w_\varphi\left(\lim_{n\to\infty}\mathfrak{u}_n,\lim_{n\to\infty}\mathfrak{x}_n\right)$$
$$=\lim_{n\to\infty}w_\varphi(\mathfrak{x}_n,\mathfrak{u}_n)=0$$

and $w_\varphi(\mathfrak{y},\mathfrak{v})=w_\varphi\left(\lim_{n\to\infty}\mathfrak{v}_n,\lim_{n\to\infty}\mathfrak{y}_n\right)$

$$=\lim_{n\to\infty}w_\varphi(\mathfrak{y}_n,\mathfrak{v}_n)=0.$$

Therefore, $\mathfrak{x}=\mathfrak{u}$ and $\mathfrak{y}=\mathfrak{v}$, which implies $(\mathfrak{x},\mathfrak{y})$ is a coupled fixed point of .

To prove the uniqueness, let us assume another coupled fixed point $(\mathfrak{x}^*,\mathfrak{y}^*)\in\mathfrak{M}^2\cup\mathfrak{V}^2$. If $(\mathfrak{x}^*,\mathfrak{y}^*)\in\mathfrak{M}^2$, then we obtain

$$w_\varphi(\mathfrak{x},\mathfrak{x}^*)=w_\varphi(\xi(\mathfrak{x}^*,\mathfrak{y}^*),\xi(\mathfrak{x},\mathfrak{y}))$$
$$\leq\mathfrak{a}(\mathfrak{x}^*,\mathfrak{x})+\mathfrak{b}(\mathfrak{y}^*,\mathfrak{y})$$

and

$$w_\varphi(\mathfrak{y},\mathfrak{y}^*)=w_\varphi(\xi(\mathfrak{y}^*,\mathfrak{x}^*),\xi(\mathfrak{y},\mathfrak{x}))$$
$$\leq\mathfrak{a}(\mathfrak{y}^*,\mathfrak{y})+\mathfrak{b}(\mathfrak{x}^*,\mathfrak{x}).$$

Thus, $w_\varphi(\mathfrak{x},\mathfrak{x}^*)+w_\varphi(\mathfrak{y},\mathfrak{y}^*)\leq\Gamma((\mathfrak{x}^*,\mathfrak{x})+(\mathfrak{y}^*,\mathfrak{y}))$ (16)

Since $\mathfrak{a}+\mathfrak{b}=\Gamma<1$, from (16), we have $w_\varphi(\mathfrak{x}^*,\mathfrak{x})+w_\varphi(\mathfrak{y}^*,\mathfrak{y})=0$.

Hence, we get $\mathfrak{x}^*=\mathfrak{x}$ and $\mathfrak{y}^*=\mathfrak{y}$. Also, if $(\mathfrak{x}^*,\mathfrak{y}^*)\in\mathfrak{V}^2$, we have $\mathfrak{x}^*=\mathfrak{x}$ and $\mathfrak{y}^*=\mathfrak{y}$. Thus, $(\mathfrak{x},\mathfrak{y})$ is a unique coupled fixed point of ξ.

The following corollary is deduced from Theorem 4.1, by assuming the constants $\mathfrak{a}=\mathfrak{b}$.

Corollary 3.6. Let $(\mathfrak{M},\mathfrak{V},w_\varphi)$ be a complete modular bipolar metric space, $\xi:(\mathfrak{M}^2,\mathfrak{V}^2)\rightrightarrows(\mathfrak{M},\mathfrak{V})$ be a covariant mapping. Suppose that the condition follows

$$w_\varphi(\xi(\mathfrak{x},\mathfrak{y}),\xi(\mathfrak{u},\mathfrak{v}))\leq\frac{\mathfrak{a}}{2}\left(\mathfrak{a}\ w_\varphi(\mathfrak{x},\mathfrak{u})+\mathfrak{b}w_\varphi(\mathfrak{y},\mathfrak{v})\right),\quad \mathfrak{a}<1$$

where $\mathfrak{x},\mathfrak{y}\in\mathfrak{M}$, $\mathfrak{u},\mathfrak{v}\in\mathfrak{V}$, then ξ has a unique coupled fixed point.

Application

Integral Equations

Theorem 4.1. Consider the integral equation

$$\varphi(\mathfrak{x})=f(\mathfrak{x})+\int_{\mathfrak{M}\cup\mathfrak{V}}P(\mathfrak{x},\mathfrak{y},\varphi(\mathfrak{y}))d\mathfrak{y} \tag{17}$$

where $\mathfrak{x}\in\mathfrak{M}\cup\mathfrak{V}$, and $\mathfrak{M}\cup\mathfrak{V}$ is a Lebesgue measurable set. Suppose that:

- $P:(\mathfrak{M}^2\cup\mathfrak{V}^2)\times[0,\infty)\to[0,\infty)$ and $f\in L^\infty(\mathfrak{M})\cup L^\infty(\mathfrak{V})$;
- $\tilde{a}:\mathfrak{M}^2\cup\mathfrak{V}^2\to[0,\infty)$ is a continuous function such that $|P(\mathfrak{x},\mathfrak{y},\varphi(\mathfrak{y}))-P(\mathfrak{x},\mathfrak{y},\psi(\mathfrak{y}))|\leq k.\gamma(\mathfrak{x},\mathfrak{y})|\varphi(\mathfrak{y})-\psi(\mathfrak{y})|$ for $(\mathfrak{x},\mathfrak{y})\in\mathfrak{M}^2\cup\mathfrak{V}^2$; $\int_{\mathfrak{M}\cup\mathfrak{V}}(\tilde{a}\ \mathfrak{y})\ \mathfrak{y}\leq$, that is, $\sup_{\mathfrak{x}\in\mathfrak{M}\cup\mathfrak{V}}\int_{\mathfrak{M}\cup\mathfrak{V}}|\tilde{a}(\mathfrak{x},\mathfrak{y})|d\mathfrak{y}\leq1$.

Then the integral equation (17) has a unique solution in $L^\infty(\mathfrak{M})\cup L^\infty(\mathfrak{V})$.

Proof.

Consider two normed linear spaces $A=L^\infty((\mathfrak{M})$ and $B=L^\infty((\mathfrak{V})$, where $\mathfrak{M}$, $\mathfrak{V}$ are Lebesgue measurable sets with $m(\mathfrak{M}\cup\mathfrak{V})<\infty$. Define

$w_\varphi:\mathfrak{M}\times\mathfrak{V}\to[0,\infty)$ by $w_\varphi(\varphi,\psi)=\varphi-\psi_\infty$, $\forall\varphi,\psi\in\mathfrak{M}\times\mathfrak{V}$.

Then $(\mathfrak{M},\mathfrak{V},w_\varphi)$ is a complete modular bipolar metric space. Define the covariant mapping $I:L^\infty(\mathfrak{M})\cup L^\infty((\mathfrak{V})\to L^\infty((\mathfrak{M})\cup L^\infty(\mathfrak{V})$ *by*

$$I(\varphi(\mathfrak{x}))=f(\mathfrak{x})+\int_{\mathfrak{M}\cup\mathfrak{V}}P(\mathfrak{x},\mathfrak{y},\varphi(\mathfrak{y}))d\mathfrak{y},$$

where $\mathfrak{x}\in\mathfrak{M}\cup\mathfrak{V}$.

Now, we have

$$w_\varphi(I(\varphi(\mathfrak{x})),I(\psi(\mathfrak{x})))=$$

$$I(\varphi(\mathfrak{x}))-I(\psi(\mathfrak{x}))_\infty$$

$$=$$

$$\left|f(\mathfrak{x})+\int_{\mathfrak{M}\cup\mathfrak{V}}P(\mathfrak{x},\mathfrak{y},\varphi(\mathfrak{y}))d\mathfrak{y}-f(\mathfrak{x})-\int_{\mathfrak{M}\cup\mathfrak{V}}P(\mathfrak{x},\mathfrak{y},\psi(\mathfrak{y}))d\mathfrak{y}\right|$$

$$\leq\int_{\mathfrak{M}\cup\mathfrak{V}}|P(\mathfrak{x},\mathfrak{y},\varphi(\mathfrak{y}))-P(\mathfrak{x},\mathfrak{y},\psi(\mathfrak{y}))|d\mathfrak{y}$$

$$\leq k\int_{\mathfrak{M}\cup\mathfrak{V}}\gamma(\mathfrak{x},\mathfrak{y})\ |\varphi(\mathfrak{y})-\psi(\mathfrak{y})|\ d\mathfrak{y}$$

$$\leq k\ \varphi-\psi_\infty\int_{\mathfrak{M}\cup\mathfrak{V}}\gamma(\mathfrak{x},\mathfrak{y})\ d\mathfrak{y}$$

$$\leq k\ \varphi-\psi_\infty\sup_{x\in\mathfrak{M}\cup\mathfrak{V}}\int_{\mathfrak{M}\cup\mathfrak{V}}|\gamma(\mathfrak{x},\mathfrak{y})|\ d\mathfrak{y}$$

$$\leq k\varphi-\psi_\infty$$

$$=k\ w_\varphi(\varphi(\mathfrak{x}),\psi(\mathfrak{x}))$$

Thus, all the conditions of Theorem 3.1 are satisfied and has a unique fixed point. Hence, the integral equation has a unique solution.

Acknowledgements

None.

References

Ahmad, H., Younis, M., & Abdou, A. A. N. (2023). Bipolar b-metric spaces in graph setting and related fixed points. *Symmetry*, *15*(6), 1227.

Aksoy, Ü., Karapınar, E., & Erhan, İ. M. (2017). Fixed point theorems in complete modular metric spaces and an application to anti-periodic boundary value problems. *Filomat*, *31*(17), 5475-5488.

Alamri, B. (2024). F-Bipolar Metric Spaces: Fixed Point Results and Their Applications. *Axioms*, *13*(9), 609.

Bhaskar, T. G., & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear analysis: theory, methods & applications*, *65*(7), 1379-1393.

Chistyakov, V. V. (2006, January). Metric modulars and their application. In *Doklady Mathematics* (Vol.73,No. 1, pp. 32-35). Moscow: Nauka/Interperiodica.

Chistyakov, V. V. (2008). Modular metric spaces generated by F-modulars. *Folia Math*, *14*(3).

Chistyakov, V. V. (2010). Modular metric spaces, I: basic concepts. *Nonlinear Analysis: Theory, Methods & Applications*, *72*(1), 1-14.

Chistyakov, V. V. (2010). Modular metric spaces, II: Application to superposition operators. *Nonlinear Analysis: Theory, Methods & Applications*, *72*(1), 15-30.

Gürdal, U. T. K. U., Mutlu, A., & Özkan, K. Ü. B. R. A. (2020). Fixed point results for α-ψ-contractive mappings in bipolar metric spaces. *J. Inequal. Spec. Funct*, *11*(1), 64-75.

Guo, D., & Lakshmikantham, V. (1987). Coupled fixed points of nonlinear operators with applications. *Nonlinear analysis: theory, methods & applications*, *11*(5), 623-632.

Mongkolkeha, C., Sintunavarat, W., & Kumam, P. (2011). Fixed point theorems for contraction mappings in modular metric spaces. *Fixed Point Theory and Applications*, *2011*(1), 93.

Mutlu, A., & Gürdal, U. (2016). Bipolar metric spaces and some fixed point theorems. *Journal of nonlinear science and applications*, *9*(9).

Mutlu, A., Ozkan, K., & Gurdal, U. (2017). Coupled fixed point theorems on bipolar metric spaces. *European journal of pure and applied Mathematics*, *10*(4), 655-667.

Mutlu, A., Yolcu, N., & Mutlu, B. (2015). Coupled fixed point theorem for mixed monotone mappings on partially ordered dislocated quasi metric spaces. *Global Journal of Mathematics Vol*, *1*(1).

Rani, J., & Prakasam, A. (2024). Fixed points of operators with multiplicative closed graphs on bipolar b-multiplicative metric spaces. *Adv. Fixed Point Theory*, *14*, Article-ID.

Rao, B. S., Kishore, G. N. V., & Kumar, G. K. (2018). Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy.

International Journal of Mathematics Trends and Technology-IJMTT, *63*.

Rawat, S., Dimri, R. C., & Bartwal, A. (2022). F-bipolar metric spaces and fixed point theorems with applications. *J. Math. Computer Sci*, *26*, 184-195.

Sumalai, P., Kumam, P., Cho, Y. J., & Padcharoen, A. (2017). The (CLR_g)-property for Coincidence Point Theorems and Fredholm Integral Equations in Modular Metric Spaces. *European Journal of Pure and Applied Mathematics*, *10*(2), 238-254.