On a Special ($\gamma, \beta$) - Metric and its Hypersurfaces

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ABSTRACT
In this paper we have considered a special ($\gamma, \beta$) – metric (1.2) and find some important tensors. We have also considered the hypersurface given by the equation $b(x) = \text{constant}$ with the special ($\gamma, \beta$) – metric (1.2).

Keywords : Finsler Space with ($\gamma, \beta$) – metric, cubic metric, one form metric, angular metric tensor, fundamental tensor and reciprocal tensor.

INTRODUCTION
In 1979 Matsumoto.M. [1] introduced the concept of cubic metric given by

$$L(x, y) = \left\{a_{ijk}(x)y_iy_jy_k\right\}^{1/3}$$

(1.1)

where $a_{ijk}(x)$ are the components of symmetric tensor field of (0, 3) type which depends on the position $x$ alone.

In 2011 Pandey. T. N. and Chaubey. V. K.[9], introduced the concept of ($\gamma, \beta$) – metric

$$L = L(\gamma, \beta)$$

where $\gamma$ is a cubic metric and $\beta = b_iy^i$ is a one form metric.

We have also some research papers on ($\gamma, \beta$) – metric [2,7,9,10,11,12], whose studies have definitely contributed to the growth of Finsler geometry. In this paper we have introduced a special ($\gamma, \beta$) – metric given by

$$L^3 = C_1\gamma^3 + 3C_2\gamma^2\beta + 3C_3\gamma\beta^2 + C_4\beta^3$$

(1.2)

By taking this metric, we have calculated basic tensors such as $l_i, h_{ij}, g_{ij}, \delta^i_j$ and some important theorems.

In 1995, Matsumoto. M., [6], had discussed the properties of a special hypersurface of Rander’s spaces with $b_i(x) = (\partial_i b)$ being the gradient of a scalar function $b(x)$. He had considered a hypersurface given by the equation $b(x) = \text{constant}$.

In this paper we have also used the hypersurface given by the equation $b(x) = \text{constant}$, of the Finsler space with a special ($\gamma, \beta$) – metric given by the equation (1.2).

2. The Finsler space with metric (1.2)
Let $F^n = (M^n, L)$ be an n –dimensional Finsler space with ($\gamma, \beta$) – metric given by the equation (1.2), where

$$\gamma = \left\{a_{ijk}(x)y_iy_jy_k\right\}^{1/3}$$

is a Cubic metric in $M^n$ and $\beta = b_iy^i$ is a 1 – form in $M^n$. The derivatives of $L(\gamma, \beta)$ with respect to $\gamma$ and $\beta$ are given by

$$L_\gamma = L^2 \left( C_1\gamma^2 + 2C_2\gamma\beta + C_3\beta^2 \right)$$

(2.1)

$$L_\beta = L^2 \left( C_2\gamma^2 + 2C_3\gamma\beta + C_4\beta^3 \right)$$

(2.2)

$$L_{\gamma\gamma} = 2L^2 \left( C_1\gamma + C_2\beta - LL_\gamma \right)$$

(2.3)

$$L_{\gamma\beta} = 2L^2 \left( C_1\gamma + C_3\beta - LL_\beta \right)$$

(2.4)

$$L_{\beta\beta} = 2L^2 \left( C_2\gamma + C_4\beta - LL_\gamma \right)$$

(2.5)

Where

$$L_\gamma = \frac{\partial L}{\partial \gamma} \quad L_\beta = \frac{\partial L}{\partial \beta}$$

$$L_{\gamma\gamma} = \frac{\partial L_\gamma}{\partial \gamma} \quad L_{\gamma\beta} = \frac{\partial L_\beta}{\partial \beta}$$

The normalized element of support $l_i = \hat{\partial}_i L$ is given by

$$l_i = \frac{L}{\gamma^2} a_i + L_\beta b_i$$

(2.6)

The angular metric tensor $h_{ij} = L\hat{\partial}_i \hat{\partial}_j L$, is obtained as

$$h_{ij} = P_2(a_i a_j + P_0b_i b_j + P_2(a_i b_j + a_j b_i)) + P_4 a_i a_j$$

(2.7)
Where \( P_0 = LL_{etaeta} = 2L^{-1}\left(C_3\gamma + C_4\beta - LL^2\beta \right) \)

\[
P_{-1} = \frac{2LL_\beta}{\gamma^2} = 2\gamma^{-2}L^{-1}\left(C_1\gamma^2 + 2C_2\gamma\beta + C_3\beta^2 \right)
\]

\[
P_{-2} = \frac{LL_{\beta\beta}}{\gamma^2} = 2\gamma^{-2}L^{-1}\left(C_2\gamma + C_3\beta - LL_\beta \right)
\]

\[
P_{-4} = \frac{L}{\gamma^4}\left(L_{\gamma\gamma} - 2\frac{L_{L_\gamma}}{\gamma} \right) = -2\gamma^{-5}L^{-1}\left(C_2\gamma\beta + C_3\beta^2 + \gamma LL^2_{\gamma} \right)
\]

In equation (2.7) the subscripts of coefficients \( P_{-1}, P_0, P_{-2} \) and \( P_{-4} \) are used to indicate respective degrees of homogeneity.

The fundamental tensor \( g_{ij} = h_{ij} + ll_{ij} \) is given by

\[ g_{ij} = P_{-1}a_i a_j + q_0 b_i b_j + q_{-2} \left(a_i b_j + a_j b_i \right) + q_{-4} a_i a_j \]  \hspace{1cm} (2.8)

If

\[ q_0 = P_0 + L^2\beta = 2L^{-1}\left(C_3\gamma + C_4\beta - \frac{1}{2}LL^2\beta \right) \]

\[ q_{-2} = P_{-2} + L_{L_\beta} = 2\gamma^{-2}L^{-1}\left(C_2\gamma + C_3\beta - LL_\beta \right) \]

\[ q_{-4} = P_{-4} + L^2_{L_\gamma} = -\gamma^{-5}L^{-1}\left(2C_2\gamma\beta + 2C_3\beta^2 + \gamma LL^2_{\gamma} \right) \]  \hspace{1cm} (2.9)

We know that \( g^{ij} g_{ij} = \delta^h_i \)

The reciprocal tensor \( g^{ij} \) of \( g_{ij} \) is given by

\[
g^{ij} = \frac{a^{ij}}{P_{-1}} - \frac{d (q_{-2} \pi_{-1} - q_{-4} \tau_{-1})}{P_{-1}} a^i a^j - \frac{d (q_{-2} \pi - q_0 \pi)}{P_{-1}} b^i b^j - \frac{d (q_0 \pi_{-1} - q_{-2} \tau_{-1})}{P_{-1}} a^i b^j - \frac{d (q_{-4} \tau - q_{-2} \pi)}{P_{-1}} a^j b^i
\]

Where

\[
P_{-1} + q_{-4} \gamma^3 + q_{-2} \beta = \pi, \quad q_{-2} \gamma^3 + q_0 \beta = \tau
\]

\[
q_{-4} \beta + q_{-2} b^2 = \pi_{-1}, \quad P_{-1} + q_{-2} \beta + q_0 b^2 = \tau_{-1}
\]

\[
d = \frac{1}{\tau_{-1} - \pi_{-1}}
\]

\[
g^{ij} = \frac{a^{ij}}{P_{-1}} - S'_0 b^i b^j - S'_1 \left(a^i b^j + a^j b^i \right) - S'_2 a^i a^j
\]  \hspace{1cm} (2.10)

Where

\[
d = \frac{1}{\left\{q_0 q_{-4} - (q_{-2})^2 \right\} \left(\beta^2 - b^2 \gamma^3 \right) - P_{-1} \left(P_{-1} + 2q_{-2} \beta + q_0 b^2 + q_{-4} \gamma^3 \right)}
\]

\[
S'_0 = \frac{d (q_{-2} \tau - q_0 \pi)}{P_{-1}} = \frac{d \left\{ (q_{-2})^2 - q_0 q_{-4} \right\} \gamma^3 - q_0 P_{-1}}{P_{-1}}
\]

\[
S'_1 = \frac{d (q_0 \pi_{-1} - q_{-2} \tau_{-1})}{P_{-1}} = \frac{d (q_{-4} \tau - q_{-2} \pi)}{P_{-1}} = \frac{d \left\{ q_0 q_{-4} - (q_{-2})^2 \right\} \beta - P_{-1} q_{-2}}{P_{-1}}
\]

\[
S'_2 = \frac{d (q_{-2} \pi_{-1} - q_{-4} \tau_{-1})}{P_{-1}} = \frac{d \left\{ (q_{-2})^2 - q_0 q_{-4} \right\} b^2 - q_{-4} P_{-1}}{P_{-1}}
\]
Theorem (2.1) The angular metric tensor $h_{ij}$, the fundamental tensor $g_{ij}$ and its reciprocal tensor $g^{ij}$ of $(\gamma, \beta)$–metric are given by equations (2.7), (2.8) and (2.10) respectively.

3. The Hypersurfaces $F^{(n-1)}(c)$

In this section, we have taken the special $(\gamma, \beta)$ – metric with a gradient $b_i(x) = \partial x^i b(x)$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ which is given by the equation $b(x) = c$ (constant).

Since the parametric equation of $F^{n-1}(c)$ is $x^i = x^i(u^a)$, hence

\[(\partial / \partial u^a) b(x(u)) = 0 = b_i(x) X^i_a, \text{ where } b_i(x) \text{ are covariant components of a normal vector field of } F^{n-1}(c)\]

Therefore, along the $F^{n-1}(c)$, we have

\[b_i X^i_a = 0 \quad \text{and} \quad b_i y^j = 0 \quad (3.1)\]

In general, the induced metric

\[L(u,v) = C_1 \left\{ a_{\alpha \beta} (u) \right\}^{\alpha \beta \gamma \gamma} \]

where

\[a_{\alpha \beta} (u) = a_{ij} (x(u)) X^i_\alpha X^j_\beta X^k_\gamma \]

\[X^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \]

By using equation (3.1) and (2.10), we have

\[g^{ij} b_j = b^2 \left( \frac{1}{P-1} - S^i_0 b^2 \right) \]

where

\[b^2 = a^{ij} b_j b_j \]

Hence, we get

\[b_i = b \sqrt{\frac{1}{P-1} - b^2 S^i_0} N_i \quad (3.3)\]

Hence from (2.10) and (3.3) we have

\[b^i = a^{ij} b_j = \frac{b}{\sqrt{\frac{1}{P-1} - b^2 S^i_0}} N^i + \frac{b^2 S^i_0}{\frac{1}{P-1} - b^2 S^i_0} a^i \quad (3.4)\]

Theorem (3.1). Let $F^n$ be a Finsler space with $(\gamma, \beta)$–metric (1.2), $b(x) = \partial x b(x)$ and $F^{n-1}(c)$ be a hypersurface of $F^n$ given by $b(x) = c$ (constant). If $b_i$ is a non-zero field, then the induced metric of $F^{n-1}(c)$ is given by (3.2) and relations (3.3) and (3.4) hold.

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