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RESEARCH ARTICLE

Fixed Point Theorems in Controlled Multiplicative Metric-Like Space

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Abstract

In this paper, we have introduced a new concept in metric spaces called Controlled Multiplicative Metric-like Spaces. We have established fixed point theorems in this space by applying contraction. Additionally, we present a non-trivial example that demonstrates the existence of fixed points within this new space. The findings contribute to the ongoing development of fixed point theory in generalized metric-like structures and open avenues for further research in related analytical and applied domains.

Mathematical Subject Classification: Primary 47H10; Secondary 54H25

Keywords: Controlled multiplicative metric-like spaces, contraction, G-continuous, Graph Connectivity, Uniqueness.

Introduction

In 1922, Banach introduced a ground breaking concept the uniqueness of fixed point theory, and he proved numerous results using this idea. Subsequently, many mathematicians built upon Banach's work. In 2018, Jachymski utilized fixed point theory concepts in the field of Graph theory. By merging fixed point theory and graph theory, he introduced a unique graphical representation. Building on this, Nabil Malaiki et al. Developed a new fixed point theory for controlled metric type spaces with graph structures. The set of positive real numbers is not complete under the usual metric. To address this, the usual metric space is modified to a multiplicative metric space, where the set of positive real numbers becomes complete. In 2008, Bashirov et al. introduced the concept of multiplicative metric spaces and established initial fixed point results within this framework. Subsequently, Ozavsar and Cevikel built upon this work, proving fixed point theorems for multiplicative contraction mappings in multiplicative metric spaces .

Extending these ideas, this paper explores multiplicative metric spaces. Specifically, it proves novel fixed point theorems in multiplicative metric spaces where the distance between two points equals one. Fixed point theorems have been developed by various researchers in multiplicative metric spaces. In this work, we have established the theorems using an "if" condition rather than an "if and only if" condition. Utilizing this approach, we have extended controlled metric-like spaces to multiplicative metric spaces.

Preliminaries

Definition 2.1[5] Let A be a non empty set, Multiplicative metric is a mapping $\varrho : A\times A \rightarrow \Re$ satisfying the following conditions:

- $\varrho(s,r)>1$ for all $s,r\in\Re$ and $\varrho(s,r)=1$ if and only if s = r
- $\varrho(s,r)=\varrho(r,s)$ for all $s,r\in\Re$
- $\varrho(s,z)=\varrho(s,r)\,\varrho(r,z)$ for all $s,r,z\in\Re$

Example 2.2 Let be a fixed number. Then $Q_a : F\times F\rightarrow[1,\infty]$ is defined by

$$Q_a(s,r)= a^{|s-r|}$$

holds the multiplicative metric conditions.

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Definition 2.3[5] In a multiplicative metric space (S,ϱ) the $\in$ -neighborhood of $s\in S$ where $\in>1$, is defined as: $B_{\in}(S)=r\in S$: $\varrho(s,r)<\in$ (open) $B_{\in}(s):Q(s,r)\leq\in$ (closed)

Definition 2.4[5] Consider (S, ϱ) a multiplicative metric space, a sequence $\{a_n\}$ in S and $s\in S$ then a_n converges multiplicatively to s. if For every multiplicative open ball $B_{\in}(s)$, there exists a natural number Q such that if $q\geq Q\Rightarrow S_q\in:B_{\in}(s):$, then the sequence $\{s_q\}$ is said to be multiplicative converging to s, denoted by $s_q\to s(p\to\infty)$.

Definition 2.5[5] A sequence $\{s_n\}$ in S is multiplicative Cauchy with respect to the multiplicative metric ϱ if: For any $\epsilon>1$, there exists $q\in Q$ such that $\varrho(s_p,s_q)<\epsilon$ whenever $p,q\geq Q$.

Definition 2.6[5] Let (A, Q) be a multiplicative metric space and $\{a_n\}$ a sequence in A. The sequence is said to be multiplicative cauchy sequence, if it is multiplicative convergent.

Definition 2.7[8] Assume $F\neq\emptyset$ and $\varrho:F\times F\to[1,\infty)$ the function $\mu:F\times F\to[0,\infty)$ if it satisfies the following conditions:

- $\varrho(s,r)=0$ if s = r
- $\varrho(s,r)=\varrho(r,s)$
- $\varrho(s,r)\leq\mu(s,z)\varrho(s,z)+\mu(z,r)\varrho(z,r)$ for all $s,r,z\in F$

is said to be controlled metric like space of F.

Result and Discussion

Definition 3.1. Assume $F\neq\emptyset$ and $\varrho:F\times F\to[1,\infty)$ the function $\mu:F\times F\to[1,\infty)$, for all $s,r,z\in F$ if it satisfies the following conditions:

- $\varrho(s,r)=1$ if s = r
- $\varrho(s,r)=\varrho(r,s)$
- $\varrho(s,r)\leq\mu(s,z)\varrho(s,z).\mu(z,r)\varrho(z,r)$ is said to be controlled multiplicative metric

like space (Controlled MMLS) on F.

Remark 3.2 Every controlled multiplicative metric type space called controlled MMLS but the converse is not always true.

Definition 3.4 Let (F,ϱ) is not a controlled MMLS, and $\{a_n\}_{n\geq0}$ be a sequence in F.

- $\{a_n\}$ is convergent to a F, if and only if $\varrho(a_n,a)=\varrho(a,a)$ Here, one write $\lim_{n\to\infty}\{a_n\}=a$
- $\{a_n\}$ is Cauchy, if and only if $\varrho(a_n,a_m)$ exists and is finite.
- (F,ϱ) is called complete if each cauchy sequence a_n there is some a ∈F such that

$$\varrho(a_n,a)=\varrho(a,a)=\varrho(a_n,a_m).$$

Definition 3.5 Let (F,ϱ) be a controlled MMLS. Let $a\in F$ and $\epsilon>1$ To define a set

$$B_{\in}(a)=\{b\in S\,|\,\varrho(a,b)<\epsilon\}$$

which is called multiplicative open ball of radius ϵ with centre a.

Definition 3.6 Let (F,ϱ) be a controlled MMLS. The mapping α : F → F is called continuous at $a\in F$ if for p > 1 there is q > 1 so that $\alpha(B(a,q))\subseteq B(\alpha(a)$ thus if α is continuous at a, then for any sequence $\{a_n\}$ converging to a, we have $\lim_{n\to\infty}\alpha a_n=a_\alpha$ that is $\lim_{n\to\infty}\varrho(\alpha a_n,a_\alpha)=\varrho(\alpha a,a_\alpha)=\varrho(\alpha a,\alpha a)$.

This result satisfies, Banach contraction principle on controlled MMLS.

Theorem 3.7

Let α be a self mapping on a complete controlled MMLS (F,ϱ) so that

$$\varrho(\alpha(a),\alpha(r))\leq\varrho(a,r)^K \quad (1)$$

for all $a,r\in F$, where $K\in(0,1)$. For $a_0\in F$. Consider $a_n=\alpha^n(a_0)$ then

$$sup_{m\geq1}lim_{n\to\infty}\frac{\mu(a_{i+1},a_{i+2})}{\mu(a_i,a_{i+1})}\mu(a_{i+1},a_m)<\frac{1}{K} \quad (2)$$

Let each $a\in F$

$$\lim_{n\to\infty}\mu(a_n,a),\lim_{n\to\infty}\mu(a,a_n) \quad (3)$$

exist and are finite. Then α possesses a unique fixed point that is $\tau\in F$. We have $\varrho(\tau,\tau)=1$.

Proof :

Let the sequence be $\{a_n\}=\alpha^n(a_0)$. Taking known equation 1, we have

$\varrho(a_n,a_{n+1})\leq\varrho(a_0,a_{n+1})^{K^n}$ for all $n\geq1$ for all integers $n<m$

$$\varrho(a_n,a_m)\leq\mu(a_n,a_{n+1})\varrho(a_n,a_{n+1})\,.\mu(a_{n+1},a_m)\varrho(a_{n+1},a_m)$$

$$\leq\mu(a_n,a_{n+1})\varrho(a_n,a_{n+1})\,.\mu(a_{n+1},a_m)\mu(a_{n+1},a_{n+2})\varrho(a_{n+1},a_{n+2})\mu(a_{n+1},a_m)$$
$$\mu(a_{n+2},a_m)\varrho(a_{n+2},a_m)$$

$$\leq\mu(a_n,a_{n+1})\varrho(a_n,a_{n+1})\varrho(a_n,a_{n+1}).\mu(a_{n+1},a_m)\mu(a_{n+1},a_{n+2})\varrho(a_{n+1},a_{n+2})$$
$$\mu(a_{n+1},a_m)\mu(a_{n+2},a_m)\mu(a_{n+2},a_m)\mu(a_{n+2},a_{n+3})\varrho(a_{n+2},a_{n+3}).$$
$$\mu(a_{n+1},a_m)\mu(a_{n+2},a_m)\mu(a_{n+3},a_m)$$

$$\leq\ldots\leq\mu(a_n,a_{n+1})\varrho(a_n,a_{n+1}).\prod_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})\varrho(a_i,a_{i+1})\,.$$
$$\prod_{k=n+1}^{m-1}\mu(a_k,a_m)\varrho(a_{m-1},a_m)$$

$$\leq\mu(a_n,a_{n+1})\varrho(a_0,a_1)^{k^n}.\prod_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})\varrho(a_0,a_1)^{k^i}.$$
$$\prod_{k=n+1}^{m-1}\mu(a_i,a_m)\varrho(a_0,a_1)^{k^{m-1}}$$

$$\leq\mu(a_n,a_{n+1})\varrho(a_0,a_1)^{k^n}.\prod_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})\varrho(a_0,a_1)^{k^i}.$$
$$\prod_{k=n+1}^{m-1}\mu(a_i,a_m)\varrho(a_0,a_1)^{k^{m-1}}$$

$$\leq\mu(a_n,a_{n+1})\varrho(a_0,a_1)^{k^n}.\prod_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})\varrho(a_0,a_1)^{k^i}.$$

Due to $\mu(a,r)\geq1$

$$\varrho(a_n,a_{n+1})\leq\mu(a_n,a_{n+1})\varrho(a_0,a_1)^{k^n}.\prod_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})\varrho(a_0,a_1)^{k^i}.$$

Let $\mathfrak{R}_p=\prod_{i=0}^{p}(\prod_{j=0}^{i}\mu(a_j,a_m))\mu(a_i,a_{i+1})^{k^i}$ then we have

$$\varrho(a_n,a_m)\leq\varrho(a_0,a_1)\Big[\mu(a_n,a_{n+1}).(r_{m-1},r_n)^{k^n}\Big] \quad (4)$$

Taking known equation (2)and verified by ration test satisfies the $\lim_{n\to\infty}\{a_n\}$ exist. Hence $\{a_n\}$ is a real Cauchy sequence. Letting $m,n\to\infty$ in the inequality (4) gets

$$\lim_{n\to\infty} \varrho\left(a_n, a_m\right) = 1 \quad (5)$$

The sequence $\{a_n\}$ is a Cauchy in (F,ϱ). This is a complete controlled multiplicative metric like space $\{a_n\}$ converges to some $a \in F$

$$\lim_{n\to\infty} \varrho(a_n, a) = \varrho(a,a) = \lim_{n,m\to\infty} \varrho\left(a_n, a_m\right) = 1 \quad (6)$$

Then $\varrho(a,a) = 1$. It is claimed that $\alpha a = a$. Applying condition of triangle inequality

$$\varrho\left(a_n, a_{n+1}\right) \le \mu\left(a_n, a_n\right)\varrho\left(a_n, a_n\right) .\mu\left(a_n, a_{n+1}\right)\varrho\left(a_n, a_{n+1}\right)$$

Taking all known equations (2), (3), (5), (6)

$$\lim_{n\to\infty} \varrho\left(a, a_{n+1}\right) = 1 \quad (7)$$

Using condition of triangle inequality and (1)

$$\varrho\left(a,\alpha(a)\right) \le \mu\left(a,a_{n+1}\right)\varrho\left(a,a_{n+1}\right) .\mu\left(a_{n+1},\alpha(a)\right)\varrho\left(a_{n+1},\alpha(a)\right)$$
$$\le \mu\left(a,a_{n+1}\right)\varrho\left(a,a_{n+1}\right) .\mu\left(a_{n+1},\alpha(a)\right)\varrho\left(a_{n+1},\alpha(a)\right)^k$$

Letting $n \to \infty$. Adding the findings (3),(7). It concludes that $\varrho\left(a,\alpha(a)\right) = 1$

i.e $\alpha(a) = a$. Assume that $\alpha(\sigma) = \sigma$ and $\alpha(m) = m$. Here,

$$\varrho(\sigma, m) = \varrho\left(\alpha(m)\right) \le \varrho\left(z, \alpha(z)\right)^k \quad (8)$$

It holds unless $\varrho(\sigma, m) = 1$ so $\sigma = m$.

Controlled Multiplicative Metric Like Spaces

The researchers discusses about controlled Multiplicative metric like spaces. It is denoted by (CMMLS) this metric would been endowed with graph. A graph is denoted by (V,E) here v is a set of vertices coinciding with F and E is the set of its edge with $\Delta \subset \mathrm{E}$. Δ is denoting diagonal of F^2 assumes that G has no parallel edges.

Definition 3.8 Let t and s be the two vertices of a graph G. It defines $q \in \mathrm{N} \cup 1$ to be the length of the path between t and s in G by a sequence $(k_i)_{i=0}^{q}$ and q + 1 distinct vertices where $k_0 = t,\ k_t = s$, and $(k_i,\ k_{i+1}) \in E(G)$ for i = 1, 2, ..., q.

Definition 3.9 Let (F,ϱ) be a complete CMMLS, endowed with a graph G.

The mapping $\alpha : F \to F$ is said to G_ϕ contraction if for all $t,s \in F$,

$$(t,s) \in E(G) \Rightarrow \left(\alpha(t),\alpha(s)\right) \in E(G) \quad (9)$$

there is $\varnothing : [1,\ \infty) \to [1,\ \infty)$

$$C\left(\alpha(t),\ \alpha^2(t)\right) \le (C\left(t,\ \alpha(t)\right))^{\phi} \quad (10)$$

for all $t \in X^\alpha$ where ϕ is non decreasing and $\left[c(t)\phi^n\right]_{n\in N} \to 1$ for all t > 1.

Lemma 3.10 Let (F,ϱ) be a complete CMMLS, equipped by a graph G. Suppose that $\alpha : F \to F$ be a G_ϕ contraction. If $t \in X^\alpha$ then there is q(t) ≥ 1 so that

$$\varrho\left(\alpha^n(t),\ \alpha^{n+1}(t)\right) \le \left(q(t)\right) \quad (11)$$

For all $n \in N$ where $q(t) = \varrho\left(t,\ \alpha(t)\right)$

Proof:

Let $t \in F^\alpha$ then $(t,\ \alpha(t)) \in E(G))$ or $(\alpha(t),\ t) \in E(G))$. Assume that $(t,\ \alpha(t)) \in E(G))$. Hence,

$$\left(\alpha^n,\ \alpha^{n+1}(t)\right) \in E(G) \quad (12)$$

for all $q \in Q$

$$\varrho\left(\alpha^n(t),\ \alpha^{n+1}(t)\right) \le (\varrho\left(\alpha^{n-1}(t),\ \alpha^n(t)\right)^{\phi}$$
$$\le (\varrho\left(\alpha^{n-2}(t),\ \alpha^{n-1}(t)\right)^{\phi}$$
$$\le (\varrho\left(t,\ \alpha(t)\right)^{\phi n}$$
$$= q(t)^{\phi n}.$$

Theorem 3.11

Let (F,ϱ,G) be a complete CMMLS, equipped by a graph G. suppose that $\alpha : F \to F$ be a G_ϕ contraction which is orbitally G - continuous. Consider the property (P) as follows for all $\{t_n\}_{n\in N}$ in F. If $t_n \to t$ and $(t_n, t_{n+1}) \in E(G)$ holds. Suppose that for each $S \in F$.

$$\lim_{n\to\infty} \mu(\alpha^i(a), \alpha^n(a) \text{ for all } i \ge 1, \lim_{n\to\infty} \mu\left(\alpha^n(a), \alpha^{n+1}(a)\right) \quad (13)$$

Exist and are finite. And if $\lim_{n\to\infty} \alpha^n(a) \to q \in F$ then

$$\varrho\left(\alpha(p), p\right) > \lim_{n\to\infty} sup \mu\left(\alpha(p), \alpha^n(a)\right)\varrho\left(\alpha(p),\ \alpha^n(a)\right)\varrho\left(\alpha(p), \alpha(p)\right) \quad (14)$$

Thus, the restriction of $\alpha[a]_G$ to $[a]_G$ possesses a fixed point. Moreover, if for every two fixed point q_1, q_2 it has $\mu(q_1,q_2) > 1$ then it has uniqueness of the fixed point.

Proof:

Let $a \in F^\alpha$ by lemma (4.1) there is $q(a) \ge 1$ so that $\varrho(\alpha^n(a), \alpha^{n+1}(a) \le \left(q(s)\right)^{\phi^n}$ then $\{\alpha^n(a)\}_{n\in N}$ converges to some $q \in F$. It is enough to prove that $\alpha^n(t)_{n\in N}$ is Cauchy.

$$\varrho(\alpha^n(a), \alpha^{n+m}(a) \le \mu(\alpha^n(a), \alpha^{n+1}(a)\varrho\left(\alpha^n(a), \alpha^{n+1}(a)\right).\mu\left(\alpha^{n+1}(a), \alpha^{n+m}(a)\right)\varrho\left(\alpha^{n+1}(a), \alpha^{n+m}(a)\right)$$
$$\le \mu(\alpha^n(a), \alpha^{n+1}(a)\varrho\left(\alpha^n(a), \alpha^{n+1}(a)\right).\mu\left(\alpha^{n+1}(a), \alpha^{n+m}(a)\right)$$
$$\mu\left(\alpha^{n+1}(a), \alpha^{n+2}(a)\right)\varrho(\alpha^{n+1}(a), \alpha^{n+2}(a).\mu\left(\alpha^{n+1}(a), \alpha^{n+m}(a)\right)\mu\left(\alpha^{n+2}(a), \alpha^{n+m}(a)\right)$$
$$\varrho\left(\alpha^{n+2}(a), \alpha^{n+m}(a)\right)$$
$$\le \mu\left(\alpha^n(a), \alpha^{n+1}(a)\right)\varrho\left(\alpha^n(a), \alpha^{n+1}(a)\right).\prod_{i=n+1}^{n+m-2}(\prod_{j=n+1}^{i} .\mu\left(\alpha^j(a), \alpha^{n+m}(a)\right)\mu\left(\alpha^i(a), \alpha^{i+1}(a)\right)$$
$$\varrho\left(\alpha^i(a), \alpha^{i+1}(a)\right).\prod_{k=n+1}^{n+m-1} \mu\left(\alpha^k(a), \alpha^{n+m}(a)\right)\varrho\left(\alpha^{n+m-1}(a), \alpha^{n+m}(a)\right)$$
$$\le \mu\left(\alpha^n(a), \alpha^{n+1}(a)\right)(q(a))^{\phi^n} .\prod_{i=n+1}^{n+m-2}(\prod_{j=n+1}^{i} \mu\left(\alpha^i(a), \alpha^{n+m}(a)\right)\mu\left(\alpha^i(a), \alpha^{i+1}(a)\right)$$
$$\left(q(s)\right)^{\phi^i} .\prod_{k=n+1}^{n+m-1} \mu\left(\alpha^k(a), \alpha^{n+m}(a)\right)(q)\left(s)\right)^{\phi^n + m-1} \mu\left(\alpha^{n+m-1}(a)\right), \alpha^{n+m}(a)\right)$$
$$\le \mu\left(\alpha^n(a), \alpha^{n+1}(a)\right)(q(s))^{\phi^n} .\prod_{i=n+1}^{n+m-1}(\prod_{j=0}^{i} \mu\left(\alpha^j(a), \alpha^{n+m}(a)\right)\mu\left(\alpha^i(t), \alpha^{i+1}(a)\right) \left(q(a)\right)^{\phi^n}$$
$$\le \prod_{j=0}^{n} \mu\left(\alpha^j(t), \alpha^{n+m}(a)\right)\mu\left(\alpha^n(a), \alpha^{n+1}(a)\right)\left(q(s)\right)^{\phi^n} \prod_{i=n+1}^{n+m-1} (\prod_{j=0}^{i} \mu\left(\alpha^j(a), \alpha^{n+m}(a)\right)$$
$$\mu\left(\alpha^i(a), \alpha^{i+1}(a)\right)\left(q(s)\right)^{\phi^i}$$
$$= \prod_{i=n}^{n+m-1} (\prod_{j=0}^{i} \mu\left(\alpha^j(a), \alpha^{n+m}(a)\right) \mu\left(\alpha^i(a), \alpha^{i+1}(a)\right)\left(q(s)\right)^{\phi^i}$$
$$= \prod_{i=n}^{n+m-1} M_i\left(q(s)\right)^{\phi^i}$$
$$M_i =$$
$$\left(\prod_{j=0}^{i} \mu\left(\alpha^j(t), \alpha^{n+m}(a)\right)\mu\left(\alpha^n(a), \alpha^{n+1}(a)\right)\right)$$
$$= \prod_{i=1}^{m} M_i^{n+i-1}\left(q(s)\right)^{\phi^n + i-1}$$

The property of ϕ and using the known result (13) we deduce that $\Pi_{i=1}^{m} M_i^{n+i-1}\left(q(s)\right)^{\phi^n+i-1}$ is convergent to 1 as $m,n \to \infty$

$$\lim_{n\to\infty} \varrho\left(\alpha^n(a), p\right) = \varrho(p,p) = \lim_{n\to\infty} \varrho\left(\alpha^n(a), \alpha^m(a)\right) = 1 \tag{15}$$

We have $\varrho(p,p)=1$.

Since $a \in F^\alpha$ that $\alpha^n(a) \in F^\alpha$ for every $q \in Q$. Suppose that $(a,\alpha(a)) \in E(G)$ by (p) we deuce that there is $(\alpha^{kn}(a), p) \in E(G)$ for all $q \in Q$. Consider the path in G as follows $S,\alpha(S)\ldots\ldots\ldots\ldots\alpha^{k1}(s), p)$ and so $p \in [a]_G$ the orbital G-continuity of α yields.

$$\lim_{n\to\infty} \varrho\left(\alpha^{kn}(a),(\alpha)p)\right) = \varrho\left((\alpha)p,(\alpha)p)\right) \tag{16}$$

Suppose that $\varrho\left((\alpha)p,(\alpha)p\right) > 1$ according to the known result

$$\varrho\left(\alpha(p),p\right) \le \mu\left(\alpha(p),\alpha^{kn}\right)\varrho\left(\alpha(p),\alpha^{kn}(a)\right).\mu\left(\alpha^n(a),p\right)\varrho\left(\alpha^{kn}(a),p\right)$$

Letting $n \to \infty$ applying the known equations (14),(15),(16), we get

$$\varrho\left(\alpha(p),p\right) \le \lim_{n\to\infty} sup\, \mu\left(\alpha(p),\alpha^{kn}(a)\right)\varrho\left(\alpha(p),\alpha(p)\right)$$

It contradicts (14).

Hence $\varrho(\alpha(p),p)=1$ so $\alpha(p)=p$ that is p is a fixed point of $\alpha[a]_G$ for its uniqueness there are two fixed points q_1 and q_2 that is $\alpha(q_1)=q_1$ and $\alpha(q_2)=q_2$

$$\begin{aligned}\varrho(q_1,q_2) &\le \mu(q_1,q_1)\varrho(q_1,q_1)\,.\mu(q_1,q_2)\varrho(q_1,q_2)\\ &\le \frac{\mu(q_1,q_1)}{1-\mu(q_1,q_2)}\varrho(q_1,q_1)\end{aligned}$$

Since $\mu > 1$ we have $\varrho(q_1,q_2) \le 1$ and so $q_1 = q_2$.

Hence the theorem is proved by its uniqueness and existence for common fixed point.

Conclusion

In this study, we have successfully established fixed point theorems in controlled multiplicative metric-like spaces, incorporating a graph structure. By utilizing the Banach Contraction Principle and G_ϕ contraction, we have extended existing results in fixed point theory to a broader framework. Furthermore, the provided examples validate our theoretical findings.

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