



ARTICLE TYPE

Some fixed point theorems for $\mathbb{F}$ contraction on b-multiplicative metric spaces

Augustine Antony L, Mary Priya Dharsini A*

Abstract

This paper establishes common fixed point theorems for two self-mappings in b-multiplicative metric spaces under a weakened condition of $\mathbb{F}$ -contraction. By utilizing a non-decreasing property instead of stronger traditional assumptions, the study extends existing results within the framework of $\mathbb{F}$ -contraction. It derives new fixed point theorems and analyzes Jungck's fixed point theorem in this context. The results demonstrate the existence and uniqueness of common fixed points for two self-mappings in complete b-multiplicative metric spaces. An illustrative example is provided to support and clarify the theoretical findings.

Keywords: Unique Fixed point, Two mappings, b-multiplicative metric space(b-MMS), $\mathbb{F}$ contraction, Non-decreasing property, Cauchy sequence, Continuous function.

Introduction

Fixed point theory is a vital area of modern mathematical analysis, with diverse applications in various mathematical fields, including topology, geometry, control theories, economic modeling and optimization. This theory focuses on self-mappings, where a mapping T has a fixed point, denoted by $x = Tx$.

Contractive conditions play a crucial role in solving fixed-point problems. The Banach contraction principle, established by Banach in 1922, is an important fixed point result. Due to its significance, numerous authors have extended and generalized this principle. Bashirov *et al.* introduced multiplicative metric spaces and established fixed point theorems. Subsequently, Ali *et al.* introduced the concept of b-multiplicative metric spaces in 2017 and

achieved new fixed point results. Wardowski *et al.* established $\mathbb{F}$ -contractions, an extension of Banach's contraction results. Recently, Rasham *et al.* introduced novel fixed point theorems on b-multiplicative metric spaces with applications to nonlinear integral and functional equations. After that, many researchers developed fixed point theory in b-multiplicative metric spaces and multiplicative metric spaces see references. Building these concepts, this study aims to investigate the existence and uniqueness of common fixed points for two self-mappings in complete b-multiplicative metric spaces. We provide illustrative examples to demonstrate the applicability of our findings and discuss various cases in the proof. Notably, the results should be determined solely based on the non-decreasing property, and a relevant example is provided to illustrate our findings.

Preliminaries

Definition 2.1

Let $s \geq 1$ be given real integer and let $\mathcal{C}$ be a non-empty set. A function $\mathcal{D} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{R}^+$ is a b-MMS if the conditions are true for all $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{C}$:

$$a_1) \mathcal{D}(\mathfrak{h}_1, \mathfrak{h}_2) = 1 \text{ iff } \mathfrak{h}_1 = \mathfrak{h}_2$$

$$a_2) \mathcal{D}(\mathfrak{h}_1, \mathfrak{h}_2) = \mathcal{D}(\mathfrak{h}_2, \mathfrak{h}_1)$$

$$a_3) \mathcal{D}(\mathfrak{h}_1, \mathfrak{h}_3) \leq \mathcal{D}((\mathfrak{h}_1, \mathfrak{h}_2) \cdot \mathcal{D}(\mathfrak{h}_2, \mathfrak{h}_3))^s$$

Here, the pair $(\mathcal{C}, \mathcal{D})$ is referred to as a b-MMS and a constant $s \geq 1$.

PG & Research Department of Mathematics, Holy Cross College (Autonomous), Affiliated to Bharathidasan University, Tiruchirapalli-620002, India.

***Corresponding Author:** Mary Priya Dharsini A, PG & Research Department of Mathematics, Holy Cross College (Autonomous), Affiliated to Bharathidasan University, Tiruchirapalli-620002, India., E-Mail: priyairudayam@gmail.com

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Definition 2.2

Let $\mathcal{F} : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ be a mapping as follows,

$a_1)$ $\mathcal{F}$ is increasing function i.e $\forall \mathfrak{X}, \mathfrak{Y} \in \mathcal{R}^+$ that is $\mathfrak{X} < \mathfrak{Y}, \mathcal{F}(\mathfrak{X}) < \mathcal{F}(\mathfrak{Y})$.

$a_2)$ $\{\mathfrak{X}_n\}$, is the sequence that is positive $(\mathfrak{X} > 0)$, $\lim_{i \rightarrow +\infty} \mathfrak{X}_n = 0$

iff $\lim_{i \rightarrow \infty} \mathcal{F}(\mathfrak{X}_n) = \infty$;

$a_3)$ the consent $k \in (0, 1)$ and $\lim_{i \rightarrow \infty} \mathfrak{X}^k \mathcal{F}(\mathfrak{X}) = 0$.

The mapping $\mathcal{N} : \mathcal{C} \rightarrow \mathcal{C}$ is called an $\mathbb{F}$ - contraction if there exists $\tau > 1$ then $\forall \mathfrak{X}, \mathfrak{Y} \in \mathcal{C}$, the inequality $\mathcal{D}(\mathcal{N}\mathfrak{X}, \mathcal{N}\mathfrak{Y}) > 1$ implies

$$\tau + \mathcal{F}(\mathcal{SD}(\mathcal{N}\mathfrak{X} + \mathcal{N}\mathfrak{Y})) \leq \mathcal{F}(\mathcal{D}(\mathfrak{X}, \mathfrak{Y})),$$

Then $\mathcal{N}$ denoted as an $\mathbb{F}$ - contraction.

Definition 2.3

Let $\{\mathfrak{X}_n\} \in \mathcal{X}$. and $(\mathfrak{X}, \mathcal{D})$ be a MMS. If it holds true for all $\varepsilon > 1$, then the sequence is called a multiplicative Cauchy sequence. And then the positive number $n \in \mathcal{N}$ so $\mathcal{D}(x_n, x_m) < \varepsilon$ for $n, m \geq \mathcal{N}$.

Definition 2.4

Let $\mathfrak{x} \in \mathfrak{X}$, and a sequence $(\mathfrak{x}_n)$ in $\mathfrak{X}$, then $(\mathfrak{X}, \mathcal{D})$ is said to be MMS. If a natural number $\mathbb{N}$ exists for each multiplicative open ball $\mathcal{B}(x_n)$, then $n \geq \mathbb{N} \Rightarrow x_n \in \mathcal{B}(x)$, and the sequence $\{x_n\}$ is called multiplicative convergent to x . It is denoted by $x_n \rightarrow x$ as $(n \rightarrow \infty)$.

Results and discussion**Definition 3.1**

Let the function $\mathcal{F} : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ be a mapping satisfying the following conditions.

$a_1)$ $\mathcal{F}$ is increasing function i.e $\forall \mathfrak{X}, \mathfrak{Y} \in \mathcal{R}^+$ such that $\mathfrak{X} < \mathfrak{Y}, \mathcal{F}(\mathfrak{X}) < \mathcal{F}(\mathfrak{Y})$.

$a_2)$ $\{\mathfrak{X}_i\}$, is the sequence that is pastive $(\mathfrak{X} > 1)$, $\lim_{i \rightarrow +\infty} \mathfrak{X}_n = 1$ iff $\lim_{i \rightarrow \infty} \mathcal{F}(\mathfrak{X}_n) = \infty$;

$a_3)$ there exists $k \in (0, 1)$ such that $\lim_{i \rightarrow \infty} \mathfrak{X}^k \mathcal{F}(\mathfrak{X}) = 1$.

The mapping $\mathcal{N} : \mathcal{C} \rightarrow \mathcal{C}$ is said to be an $\mathbb{F}$ - contraction if there exists $\tau > 1$ such

that for all $\mathfrak{X}, \mathfrak{Y} \in \mathcal{C}$, the inequality $\mathcal{D}(\mathcal{N}\mathfrak{X}, \mathcal{N}\mathfrak{Y}) > 1$ implies

$$\tau \cdot \mathcal{F}(\mathcal{D}(\mathcal{N}\mathfrak{X}, \mathcal{N}\mathfrak{Y}))^S \leq \mathcal{F}(\mathcal{D}(\mathfrak{X}, \mathfrak{Y}))$$

Then $\mathcal{N}$ is called an $\mathbb{F}$ - contraction.

Theorem 3.1

Let Γ be a self-map on a complete b-MMS $(\mathcal{C}, \mathcal{D})$, if Γ is continuous. If and only if the constant $\alpha \in [0, 1)$ and the map $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ is commutes with Γ and fulfills the below conditions

$$\Psi(\mathcal{C}) \subseteq \Gamma(\mathcal{C}) \text{ and } \mathcal{D}(\Psi(a), \Psi(b)) \leq \mathcal{D}(\Gamma(a), \Gamma(b))^\alpha$$

For all $a, b \in \mathcal{C}$.

Truly, the fixed point that unites Γ and Ψ is unique, if (1) holds.

Proof. Let $\Psi(\mathfrak{A}) = \mathfrak{A}$ for some $\mathfrak{A} \in \mathcal{C}$.

Assume $\Gamma : \mathcal{C} \rightarrow \mathcal{C}$ by $\Gamma(a) = \mathfrak{A}$ every $a \in \mathcal{C}$.

Then $\Gamma(\Psi(a)) = \mathfrak{A}$ and

$$\Psi(\Gamma(a)) = \Psi(\mathfrak{A}) = \mathfrak{A} (a \in \mathcal{C}), \text{ so}$$

$\Gamma(\Psi(a)) = \Psi(\Gamma(a))$ every $a \in \mathcal{C}$, and Γ commutes with Ψ , then $\Gamma(a) = \mathfrak{A} = \Psi(\mathfrak{A})$ for all $a \in \mathcal{C}$

so that $\Gamma(\mathcal{C}) \subset \Psi(\mathcal{C})$. For any $\alpha \in [0, 1)$ we get all $a, b \in \mathcal{C}$:

$$\mathcal{D}(\Gamma(a), \Gamma(b)) = \mathcal{D}(\mathfrak{A}, \mathfrak{A}) = 1 \leq \mathcal{D}(\Psi(a), \Psi(b))^\alpha.$$

If the condition (2) is true then (1) holds.

Conversely, consider a mapping Γ of $\mathcal{C}$ into itself and it commutes with Ψ if (1) holds.

To conclude that Ψ and Γ have a unique fixed point.

Let $a_0 \in \mathcal{C}$ and let a_1 be $\Psi(a_1) = \Gamma(a_0)$. Take a_p as $\Psi(a_p) = \Gamma(a_{p-1})$. We take this because $\Gamma(\mathcal{C}) \subset \Psi(\mathcal{C})$. The connection between (1) and (2) implies that $\mathcal{D}(\Psi(a_{p+1}), \Psi(a_p)) \leq \mathcal{D}(\Psi(a_p), \Psi(a_{p-1}))^\alpha$ for all p .

Take $g \in \mathcal{C}$ by (3) then $\Psi(a_p) \rightarrow g$. Actually (2) implies that (4) so we get $\Gamma(a_p) \rightarrow g$.

We know that Ψ is continuous, so (1) gives the result that Ψ and Γ are continuous. We prove that (2) and (3) have demand that $\Gamma(\Psi(a_p)) \rightarrow \Gamma(g)$, and

$\Psi(\Psi(g)) = \Psi(\Gamma(g)) = \Gamma(\Gamma(g))$ by commutativity. To prove this

$$\mathcal{D}(\Gamma(g), \Gamma(\Gamma(g))) \leq \mathcal{D}(\Psi(g), \Psi(\Gamma(g)))^\alpha = \mathcal{D}(\Gamma(g), \Gamma(\Gamma(g)))^\alpha$$

Hence, $\mathcal{D}(\Gamma(g), \Gamma(\Gamma(g)))^{(1-\alpha)} \leq 1$. So $\alpha \in [0, 1)$,

$$\Gamma(g) = \Gamma(\Gamma(g)) = \Psi(\Gamma(g));$$

i.e., $\Gamma(g)$ is the fixed point of Ψ and Γ .

assume that Ψ and Γ have unique common fixed point.

Assume that $a = \Psi(a) = \Gamma(a)$ and $b = \Gamma(b) = \Gamma(b)$.

And (1) shows

$$\mathcal{D}(a, b) = \mathcal{D}(\Gamma(a), \Gamma(b)) \leq \mathcal{D}(\Psi(a), \Psi(b))^\alpha = \mathcal{D}(a, b)^\alpha,$$

or $\mathcal{D}(a, b)(1 - \alpha) \leq 1$. Since $\alpha < 1, a = b$.

Lemma 3.1

Assume Γ and Ψ be mapping oneself on a b-MMS $(\mathcal{C}, \mathcal{D})$ and the constant $\mathcal{S} > 1$. Then $\alpha, \beta, \gamma, \delta, \varrho \in [0, 1)$

with $\alpha + \beta + \delta\gamma + \delta\delta + \varrho < 1$ then

$$\mathcal{D}(\Gamma(a_1), \Psi(a_2)) \leq \mathcal{D}(a_1, \Gamma(a_1))^\alpha \cdot \mathcal{D}(a_2, \Psi(a_2))^\beta.$$

$$\mathcal{D}(a_1, \Psi(a_2))^\gamma \cdot \mathcal{D}(a_2, \Gamma(a_1))^\delta \cdot \mathcal{D}(a_1, a_2)^\varrho,$$

for all $a_1, a_2 \in \mathcal{C}$.

Then

$$\mathcal{D}(\Gamma(a), \Psi(\Gamma(a))) \leq \mathcal{D}(a, \Gamma(a))^{\frac{\alpha + \delta\gamma + \varrho}{1 - \delta\gamma - \beta}}$$

$$(\Psi(b), \Gamma(\Psi(b))) \leq \mathcal{D}(b, \Psi(b))^{\frac{\beta + \delta\delta + \varrho}{1 - \delta\delta - \alpha}}$$

for all $a \in \mathcal{C}$ and $b \in \Gamma(\mathcal{C})$

Proof.

Consider $a \in \mathcal{C}$. then

$$\mathcal{D}(\Gamma(a), \Psi(\Gamma(a))) \leq \mathcal{D}(a, \Gamma(a))^\alpha \cdot \mathcal{D}(\Gamma(a), \Psi(\Gamma(a)))^\beta$$

Adding $\mathcal{D}(a, \Gamma(a))^{-\delta\gamma}$ to both sides of (1) and applying condition (iii) of Definition (3.1) yields

$$\begin{aligned} & \mathcal{D}(\Gamma(a), \Psi(\Gamma(a))) \cdot \mathcal{D}(a, \Gamma(a))^{-\delta\gamma} \\ & \leq \mathcal{D}(a, \Gamma(a))^\alpha \cdot \mathcal{D}(\Gamma(a), \Psi(\Gamma(a)))^\beta \cdot \mathcal{D}(a, \Psi(\Gamma(a)))^\gamma \\ & \cdot \mathcal{D}(a, \Gamma(a))^{-\delta\gamma} \cdot \mathcal{D}(a, \Gamma(a))^\varrho \\ & \leq \mathcal{D}(a, \Gamma(a))^\alpha \cdot \mathcal{D}(\Gamma(a), \Psi(\Gamma(a)))^\beta. \end{aligned}$$

$$\mathcal{D}(\Gamma(a), \Psi(\Gamma(a)))^{\delta\gamma} \cdot \mathcal{D}(a, \Gamma(a))^\varrho.$$

Thus

Let $b \in \Gamma(\mathcal{C})$. Take that

$$\mathcal{D}(\Psi(b), \Gamma(\Psi(b))) = \mathcal{D}(\Gamma(\Psi(b)), \Psi(b))$$

By adding $\mathcal{D}(\Psi(b), \Gamma(\Psi(b)))^{-\delta\delta}$ on both sides of (2),

$$\begin{aligned} & \mathcal{D}(\Psi(b), \Gamma(\Psi(b))) \cdot \mathcal{D}(\Psi(b), \Gamma(\Psi(b)))^{-\delta\delta} \\ & \leq \mathcal{D}(\Psi(b), \Gamma(\Psi(b)))^\alpha \cdot \mathcal{D}(b, \Psi(b))^\beta. \end{aligned}$$

$$\mathcal{D}(b, \Psi(b))^{\delta\delta} \cdot \mathcal{D}(b, \Psi(b))^\varrho$$

Thus

$$\mathcal{D}(\Psi(b), \Gamma(\Psi(b))) \leq \mathcal{D}(b, \Psi(b))^{\frac{\beta + \delta\delta + \varrho}{1 - \delta\delta - \alpha}}.$$

Theorem 3.2

Let $\Gamma, \Psi: \mathcal{C} \rightarrow \mathcal{C}$ be two mappings on a complete b-MMS $(\mathcal{C}, \mathcal{D})$ with $s \geq 1$. Assume that $\mathcal{F}: \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is a increasing function then $\tau > 1, 0 < \alpha, \beta < 1$ and $0 \leq \gamma, \delta, \varrho < 1$ fulfills the conditions:

(1) $\alpha + \beta + 2s\gamma + \varrho < 1$ and $\alpha + \beta + 2s\delta + \varrho < 1$,

(2) $S^2 < \frac{1}{xy}$ and $\frac{s\alpha}{1 - \delta} < 1$, and

(3) for $a, b \in \mathcal{C}$, the inequality $\mathcal{D}(\Gamma(a), \Psi(g)) > 0$ implies

$$\begin{aligned} & \tau \cdot \mathcal{F}(\mathcal{D}(\Gamma(a), \Psi(b))) \leq \mathcal{F}(\mathcal{D}(a, \Gamma(b))^\alpha \cdot \mathcal{D}(b, \Psi(b))^\beta \\ & \cdot \mathcal{D}(a, \Psi(b))^\gamma \cdot \mathcal{D}(b, \Gamma(a))^\delta \cdot \mathcal{D}(a, b)^\varrho). \end{aligned}$$

Then Γ and Ψ have a common unique fixed point.

Proof. Let a be an element of $\mathcal{C}$. Take $a_0 = a_1$ for every $n \in \mathcal{N}$ we take $a_{2n-1} = \Gamma(a_{2n-2})$ and $a_{2n} = \Psi(a_{2n-1})$.

We prove the four cases:

Case I.

Consider $a_0 = a$, this is equation is written as $a_0 = \Gamma(a_0)$, and $a_0 = \Psi(a_0)$.

Truly, if $a_0 \neq \Psi(a_0) = \Psi(\Gamma(a_0)) = \Psi(a_1)$, then $\mathcal{D}(\Gamma(a_0), \Psi(a)) > 1$. From condition (3), as it follows that

$$\begin{aligned} & \mathcal{F}(\mathcal{D}(\Gamma(a_0), \Psi(a_1))) < \tau \cdot \mathcal{F}(\mathcal{D}(\Gamma(a_0), \Psi(a_1))) \\ & \leq \mathcal{F} \cdot \mathcal{D}(a_0, \Gamma(a_0))^\alpha \cdot \mathcal{D}(a_1, \Psi(a_1))^\beta \\ & \cdot \mathcal{D}(a_1, \Gamma(a_0))^\delta \cdot \mathcal{D}(a_0, a_1)^\varrho \\ & = \mathcal{F}(\mathcal{D}(a_1, \Psi(a_1))^\beta \cdot \mathcal{D}(a_0, \Psi(a_1)))^\gamma \end{aligned}$$

Given $\mathcal{F}$ is non-decreasing, so we get

$$(\Gamma(a_0), \Psi(a_1))^{(1 - \beta - \delta\gamma)} \leq 1$$

This brings $1 - \beta - \delta\gamma \leq 1$, contradiction. Here, Γ and Ψ got a_0 as a unique point of the mappings.

Case II. Suppose $a_1 = a_2$, that is $a_1 = \Psi(a_1)$, then $a_1 = \Gamma(a_1)$.

If $a_1 \neq \Gamma(a_1) = \Gamma(\Psi(a_1)) = \Gamma(a_2)$, then

$\mathcal{D}(\Psi(a_1), \Gamma(a_2)) > 1$. From condition (3) additionally, since $\mathcal{F}$ is non-decreasing, for this is the case we have

$$\mathfrak{D}(\Gamma(a_2), \Psi(a_1)) \leq \mathfrak{D}(a_2, \Gamma(a_2))^\alpha \cdot \mathfrak{D}(a_1, \Psi(a_1))^\beta \cdot \mathfrak{D}(a_2, \Psi(a_1))^\gamma$$

$$\mathfrak{D}(a_1, \Gamma(a_2))^\delta \cdot \mathfrak{D}(a_2, a_1)^\varrho$$

By known lemma 3.1, we obtain

$$\mathfrak{D}(\Gamma(a_2), \Psi(a_1))^{(1-\alpha-\delta\delta)} \leq 1$$

Which suggests $\mathfrak{D}(\Gamma(a_2), \Psi(a_1)) = 1$, a contradiction. Consequently, a_1 is a fixed point of Γ and Ψ .

Case III. In the same way, if $a_i = a_{i+1}$, for some n , additionally, we see that a_i is a common fixed point between Γ and Ψ .

Case IV. if $a_0 \neq a_1$ and $a_1 \neq a_2$, next, we obtain from condition (1), we obtain

$$\alpha + \beta + \delta\gamma + \delta\delta + \varrho < 1$$

Taking the results of lemma 3.1 shows, we conclude that

$$\mathfrak{D}(a_1, a_2) \leq \mathfrak{D}(a_0, f(a_0))^k$$

$$\mathfrak{D}(a_2, a_3) \leq \mathfrak{D}(a_1, a_2)^i \leq \mathfrak{D}(a_0, f(a_0))^{ki}$$

$$\text{Where } k = \frac{\alpha + \delta\gamma + \varrho}{1 - \delta\gamma - \beta}, i = \frac{\beta + \delta\delta + \varrho}{1 - \delta\delta - \alpha} \text{ and } 0 < k, i < 1.$$

When this procedure is repeated, it is easy to observe that

$$\mathfrak{D}(a_i, a_{i+1}) \leq \begin{cases} \mathfrak{D}(a_0, f(a_0))^{k^{\frac{i}{2}} h^{\frac{i}{2}}}, & \text{if } i \text{ is even} \\ \mathfrak{D}(a_0, f(a_0))^{k^{\frac{i+1}{2}} h^{\frac{i-1}{2}}}, & \text{if } i \text{ is odd} \end{cases}$$

We'll now confirm that $(a_i)_{i \in \mathbb{N}}$ is Cauchy sequence.

Let $u = i + h$, if i is odd and $h > 2$, as follows

$$\begin{aligned} \mathfrak{D}(a_i, a_u) &\leq \mathfrak{D}(a_i, a_{i+1}) \cdot \mathfrak{D}(a_{i+1}, a_u)^\delta \\ &\leq \mathfrak{D}(a_i, a_{i+1})^\delta \cdot \mathfrak{D}(a_{i+1}, a_{i+2})^{s^2} \cdot \mathfrak{D}(a_{i+2}, a_u)^{s^2} \\ &\leq \mathfrak{D}(a_i, a_{i+1})^\delta \cdot \mathfrak{D}(a_{i+1}, a_{i+2})^{s^2} \cdot \mathfrak{D}(a_{i+2}, a_{i+3})^{s^3} \cdot \end{aligned}$$

$$\begin{aligned} &\mathfrak{D}(a_{i+3}, a_u)^{s^3} \\ &\leq \mathfrak{D}(a_i, a_{i+1})^\delta \cdot \mathfrak{D}(a_{i+1}, a_{i+2})^{s^2} \cdot \mathfrak{D}(a_{i+2}, a_{i+3})^{s^3} \cdot \end{aligned}$$

$$\begin{aligned} &\mathfrak{D}(a_{i+3}, a_{i+4})^{s^4} \\ &\cdot \mathfrak{D}(a_{i+4}, a_u)^{s^{u-i-1}} \cdot \mathfrak{D}(a_{i+3}, a_u)^{s^{u-i-1}} \end{aligned}$$

When n is even in this instance, we get

$$\mathfrak{D}(a_i, a_u)$$

$$\leq \mathfrak{D}(a_0, a_1)^s k^{\frac{i}{2}} h^{\frac{i}{2}} \cdot \mathfrak{D}(a_0, a_1)^{s^2} k^{\frac{i+2}{2}} h^{\frac{i+2}{2}}.$$

$$\mathfrak{D}(a_0, a_1)^{s^3} k^{\frac{i+2}{2}} h^{\frac{i+2}{2}} \cdot \mathfrak{D}(a_0, a_1)^{s^4} k^{\frac{i+4}{2}} h^{\frac{i+4}{2}} \dots$$

$$\dots \mathfrak{D}(a_0, a_1)^{s^{u-i-1} k^{\frac{u-1}{2}} h^{\frac{u-1}{2}} + k^{\frac{u-1}{2}} h^{\frac{u-1}{2}}}$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i}{2}} \left(1 + \delta k + \delta^2 k h + \delta^3 k^2 h^2 + \delta^4 k^3 h^3 + \dots \right.}$$

$$\left. + \delta^{u-i-2} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right) + \left[\delta k + \delta^3 k^2 h + \delta^5 k^3 h^2 \dots \delta^{u-i-2} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right]$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i}{2}} \left[\frac{1 - \delta^{u-i+1} k^{\frac{u-i+1}{2}} h^{\frac{u-i+1}{2}}}{1 - \delta k p} + \frac{\delta k \left(1 - \delta^{u-i-1} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right)}{h^{\frac{u-i-1}{2}}} + 1 - \delta k h \right]}$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i}{2}} \left[\frac{1 - \delta^{h+1} k^{\frac{h+1}{2}} h^{\frac{h+1}{2}}}{1 - s^2 p q} + \frac{\delta k \left(1 - \delta^{h-1} k^{\frac{h-1}{2}} h^{\frac{h-1}{2}} \right)}{1 - s^2 p q} \right]}$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i}{2}} \left[\frac{1 + S k - (1 + S k) S^h k^{\frac{h+1}{2}} h^{\frac{h+1}{2}}}{1 - s^2 p q} \right]}$$

When i is odd, we get

$$\mathfrak{D}(a_i, a_u) \leq \mathfrak{D}(a_0, a_1)^s k^{\frac{i+1}{2}} h^{\frac{i-1}{2}} \cdot \mathfrak{D}(a_0, a_1)^{s^2} k^{\frac{i+2}{2}} h^{\frac{i+2}{2}}.$$

$$\mathfrak{D}(a_0, a_1)^{s^3} k^{\frac{i+3}{2}} h^{\frac{i+1}{2}} \cdot \mathfrak{D}(a_0, a_1)^{s^4} k^{\frac{i+3}{2}} h^{\frac{i+3}{2}} \dots$$

$$\dots \mathfrak{D}(a_0, a_1)^{s^{u-i-1} k^{\frac{u-2}{2}} h^{\frac{u-2}{2}} + k^{\frac{u-2}{2}} h^{\frac{u-2}{2}}}$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i-1}{2}} \left(1 + \delta k + \delta^2 k h + \delta^3 k^2 h^2 + \delta^4 k^3 h^3 + \dots \right.}$$

$$\left. + \delta^{u-i-2} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} + \delta^{u-i-2} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right) + \left[\delta k + \delta^3 k^2 h + \delta^5 k^3 h^2 \dots \delta^{u-i-2} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right]$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i-1}{2}} \left[\frac{1 - \delta^{u-i+1} k^{\frac{u-i+1}{2}} h^{\frac{u-i+1}{2}}}{1 - s^2 k p} + \frac{\delta k \left(1 - \delta^{u-i-1} k^{\frac{u-i-1}{2}} h^{\frac{u-i-1}{2}} \right)}{1 - s^2 k h} \right]}$$

$$= \mathfrak{D}(a_0, a_1)^{s^{k^2} h^{\frac{i-1}{2}} \left[\frac{1 - \delta^{h+1} k^{\frac{h+1}{2}} h^{\frac{h+1}{2}}}{1 - s^2 k h} + \frac{\delta k \left(1 - \delta^{h-1} k^{\frac{h-1}{2}} h^{\frac{h-1}{2}} \right)}{1 - s^2 p q} \right]}$$

$$= \mathfrak{D}(a_0, a_1)^{sk^{\frac{i+1}{2}} h^{\frac{i-1}{2}}} \left[\frac{1 + \delta k - (1 + \delta k) \delta^h k^{\frac{h+1}{2}} h^{\frac{h+1}{2}}}{1 - s^2 pq} \right]$$

When h is even $h > 2$, using similar logic, we conclude that,

$$\begin{aligned} & \mathfrak{D}(a_i, a_u) \\ & \leq \mathfrak{D}(a_0, a_1)^S k^{\frac{i}{2}} h^{\frac{i}{2}} \cdot \mathfrak{D}(a_0, a_1)^{S^2} k^{\frac{i+2}{2}} h^{\frac{i}{2}} \\ & \mathfrak{D}(a_0, a_1)^{S^3} k^{\frac{i+2}{2}} h^{\frac{i+2}{2}} \cdot \mathfrak{D}(a_0, a_1)^{S^4} k^{\frac{i+4}{2}} h^{\frac{i+2}{2}} \dots \\ & \dots \mathfrak{D}(a_0, a_1)^{S^{u-i-1}} k^{\frac{u-2}{2}} h^{\frac{u-2}{2}} + \frac{u}{k^2 h^2} \\ & \leq \mathfrak{D}(a_0, a_1)^{S^{\frac{i}{2}} h^{\frac{i}{2}}} \left(1 + \delta k + \delta^2 k h + \delta^3 k^2 h^2 + \delta^4 k^3 h^3 + \dots \right. \\ & \quad \left. + \delta^{u-i-2} k^{\frac{u-i-2}{2}} h^{\frac{u-i-2}{2}} + \delta^{u-i-2} k^{\frac{u-i}{2}} h^{\frac{u-i-2}{2}} \right) \\ & \leq \mathfrak{D}(a_0, a_1)^{S k^{\frac{i}{2}} h^{\frac{i}{2}}} \left(1 + S^2 k h + S^4 k^2 h^2 + \dots + S^{u-i-2} k^{\frac{u-i-2}{2}} h^{\frac{u-i-2}{2}} \right. \\ & \quad \left. + S k + S^3 k^2 h + S^5 k^3 h^2 \dots S^{u-i-2} k^{\frac{u-i}{2}} h^{\frac{u-i-2}{2}} \right) \\ & \leq \mathfrak{D}(a_0, a_1)^{S k^{\frac{i}{2}} h^{\frac{i}{2}}} (1 + S k) \frac{1 - S^{u-i} p^{\frac{u-i}{2}} h^{\frac{u-i}{2}}}{1 - S^2 k h} \\ & \leq \mathfrak{D}(a_0, a_1)^{\delta k^{\frac{i}{2}} h^{\frac{i}{2}}} (1 + s k) \frac{1 - s^h p^{\frac{h}{2}} h^{\frac{h}{2}}}{1 - s^2 k h} \end{aligned}$$

When i and u are even,

$$\begin{aligned} & \mathfrak{D}(a_i, a_u) \\ & \leq \mathfrak{D}(a_0, a_1)^{sk^{\frac{i+1}{2}} h^{\frac{i-1}{2}}} \cdot \mathfrak{D}(a_0, a_1)^{s^2 k^{\frac{i+1}{2}} h^{\frac{i+1}{2}}} \\ & \mathfrak{D}(a_0, a_1)^{s^3} k^{\frac{i+3}{2}} h^{\frac{i+1}{2}} \cdot \mathfrak{D}(a_0, a_1)^{s^4} k^{\frac{i+3}{2}} h^{\frac{i+3}{2}} \dots \\ & \dots \mathfrak{D}(a_0, a_1)^{s^{u-i-1}} k^{\frac{u-3}{2}} h^{\frac{u-3}{2}} + k^{\frac{u-1}{2}} h^{\frac{u-1}{2}} \\ & \leq \mathfrak{D}(a_0, a_1)^{S k^{\frac{i+1}{2}} h^{\frac{i+1}{2}}} (1 + S k) \frac{1 - S^{u-i} p^{\frac{u-i}{2}} h^{\frac{u-i}{2}}}{1 - S^2 k i} \\ & \leq \mathfrak{D}(a_0, a_1)^{S k^{\frac{i+1}{2}} h^{\frac{i+1}{2}}} (1 + S k) \frac{1 - S^h p^{\frac{h}{2}} i^{\frac{h}{2}}}{1 - S^2 k i} \end{aligned}$$

When i is odd and u is even

Letting $i, u \rightarrow \infty$, we get that $\mathfrak{D}(a_i, a_u) \rightarrow 1$, since $0 \leq k, h < 1$. Therefore we conclude that (e_i) is a sequence and it is Cauchy. Then there exists $v \in \mathcal{C}$ such that

$$\lim_{i \rightarrow \infty} \mathfrak{D}(a_i, v) = 1$$

Moreover, from condition (3) of definition 3.1, we get this

$$\mathfrak{D}(\Gamma(v), v) \leq [\mathfrak{D}(\Gamma(v), a_{2i}) \cdot \mathfrak{D}(a_{2i}, v)]^S.$$

Then

$$\mathfrak{D}(\Gamma(v), v)^{\frac{1}{\delta}} \leq \limsup_{i \rightarrow \infty} \mathfrak{D}(\Gamma(v), a_{2i}) \leq \mathfrak{D}(\Gamma(v), v)^{\frac{\alpha}{1 - \delta \delta}}.$$

Hence

$$\Gamma(v) = \Psi(v) = v$$

For the uniqueness,

$$\begin{aligned} \tau \cdot \mathcal{F}(\mathfrak{D}(v, v')) &= \tau \cdot \mathcal{F}(\mathfrak{D}(\Gamma(v), \Psi(v'))) \\ &\leq \mathcal{F}(\mathfrak{D}(v, \Gamma(v))^\alpha \cdot (\mathfrak{D}(v', \Psi(v'))^\beta \\ &\quad \cdot \mathfrak{D}(v, \Psi(v')^\gamma \cdot \mathfrak{D}(v', \Gamma(v))^\delta \cdot \mathfrak{D}(v, v')^\varrho \\ &= \mathcal{F}(\mathfrak{D}(v, \Psi(v'))^\gamma \cdot \mathfrak{D}(v', \Gamma(v))^\delta \cdot \mathfrak{D}(v, v')) \end{aligned}$$

Since $\mathcal{F}$ is non-decreasing,

$$\mathfrak{D}(v, v') \leq \mathfrak{D}(v, \Gamma(v'))^\gamma \cdot \mathfrak{D}(v', \Gamma(v))^\delta \cdot \mathfrak{D}(v, v')^\varrho,$$

Which implies

$$\mathfrak{D}(v, v')^{(1 - \varrho - \gamma - \delta)} \leq 1.$$

Then

$$1 - \varrho - \gamma - \delta \leq 1$$

Which is a contradiction. Hence, $v = v'$.

Example 3.1

let $\mathcal{C} = [0, 8]$, and $\Gamma, \Psi: \mathcal{C} \rightarrow \mathcal{C}$ be two mappings stated by

$$\Gamma(a) = \begin{cases} 7, & \text{if } a \in (0, 8] \\ 8, & \text{if } a = 0 \end{cases}$$

and

$$\Psi(a) = \begin{cases} 7, & \text{if } a \in (0, 8] \\ 6, & \text{if } a = 0 \end{cases}$$

We state that a b -multiplicative metric $\mathfrak{D}: \mathcal{C} \times \mathcal{C} \rightarrow [0, \infty)$ by

$$\mathfrak{D}(a, b) = e^{(a-b)^2}, \text{ for all } a, b \in \mathcal{C}.$$

Obviously, $(\mathcal{C}, \mathfrak{D})$ is a complete b -multiplicative metric space with constant $s = 2$. We see that $\mathfrak{D}(\Gamma(a), \Psi(b)) > 1$,

when

$$(a, b) \in \Gamma = \{(a, b): a \in (0, 8], b = 0\} \cup \{(a, b): a = 0, b \in (0, 8]\} \cup \{(a, b): a = 0, b = 0\}.$$

Define $\mathcal{U}: \mathcal{C} \times \mathcal{C} \rightarrow [0, \infty)$ by

$$\begin{aligned} \tau \cdot \mathcal{F}(\mathfrak{D}(\Gamma(a), \Psi(b))) \\ \leq \mathcal{F} \left(\mathfrak{D}(a, \Gamma(a))^\alpha \cdot \mathfrak{D}(b, \Psi(b))^\beta \cdot \mathfrak{D}(a, \Psi(b))^\gamma \cdot \mathfrak{D}(b, \Gamma(a))^\delta \cdot \mathfrak{D}(a, b)^\varrho \right)^\varrho \end{aligned}$$

Substitute all the values in the above formula

$$\mathcal{U}(a, b) = \mathfrak{D}(a, \Gamma(a))^{\frac{1}{4}} \cdot \mathfrak{D}(b, \Psi(b))^{\frac{1}{4}} \cdot \mathfrak{D}(a, b)^{\frac{1}{16}}, a, b \in \mathcal{C}.$$

We take three cases to verify the formula

(i) If $a \in (0, 8]$ and $b = 0$, then we get

$$\begin{aligned} \frac{1}{6} - \frac{1}{\mathfrak{D}(\Gamma(a), \Psi(0)) + 1} &\leq \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} \\ &< -\frac{4}{36} = -\frac{4}{\mathfrak{D}(0, \Psi(0))} \\ &\leq -\frac{1}{\mathcal{U}(a, 0)} \\ &< -\frac{1}{\mathcal{U}(a, 0) + 1} \end{aligned}$$

(ii) If $b \in (0, 8]$ and $a = 0$, we get

$$\begin{aligned} \frac{1}{6} - \frac{1}{\mathfrak{D}(\Gamma(0), \Psi(b)) + 1} &\leq \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} \\ &< -\frac{4}{64} = -\frac{4}{\mathfrak{D}(0, \Psi(0))} \\ &\leq -\frac{1}{\mathcal{U}(0, b)} \\ &< -\frac{1}{\mathcal{U}(0, b) + 1} \end{aligned}$$

(iii) If $b = 0$ and $a = 0$, we get

$$\begin{aligned} \frac{1}{6} - \frac{1}{\mathfrak{D}(\Gamma(0), \Psi(0)) + 1} &\leq \frac{1}{6} - \frac{1}{3} \\ &< -\frac{1}{25} \\ &\leq -\frac{1}{\mathcal{U}(0, 0)} \\ &< -\frac{1}{\mathcal{U}(0, 0) + 1} \end{aligned}$$

Taking all the values

$$\frac{1}{6} - \frac{1}{\mathfrak{D}(\Gamma(a), \Psi(b)) + 1} < -\frac{1}{\mathcal{U}(a, b) + 1}$$

for all (a, b) .

Consequently, if we take into $\mathcal{F}(h) = -\frac{1}{h+1}$, where

$h \geq 0$, $b \in (0, \infty)$, and $\tau = \frac{1}{6}$, every conditions of theorem

(3.2) are fulfilled with $\alpha = \frac{1}{4}, \beta = \frac{1}{4}, \gamma = 0, \delta = 0, \varrho = \frac{1}{6}$. In

the meantime, we observe that $\Gamma(r) = \Psi(r) = r$ iff $r = 7$.

Hence Γ and Ψ have single fixed point.

Theorem 3.3

Let Γ and Ψ be two self-mapping on a complete b-MMS($\mathcal{C}, \mathfrak{D}$) with constant $\delta \geq 1$. Let

$\alpha, \beta \in (0, 1), \gamma, \delta, \varrho \in [0, 1]$ be the positive numbers with

$$\alpha + \beta + \delta\gamma + \delta\delta + \varrho < 1,$$

$$\delta^2\alpha + \delta^2\delta + (\delta^3 + \delta^4)(\gamma + \varrho) < 1,$$

$$\text{and } \delta^2\beta + \delta^2\gamma + (\delta^3 + \delta^4)(\delta + \varrho) < 1.$$

Define $\sigma: [0, 1] \times [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ by

$$\sigma(\alpha, \beta, \gamma, \delta, \varrho) = \min \left\{ \frac{1 - \delta^2\alpha - \delta^2\beta - (\delta^3 \cdot \delta^4)(\gamma \cdot \varrho)}{1 - \delta^3\gamma - \delta^2\delta - \delta^3\varrho}, \frac{1 - \delta^2\beta - \delta^2\gamma - (\delta^3 \cdot \delta^4)(\delta \cdot \varrho)}{1 - \delta^2\gamma - \delta^3\delta - \delta^3\varrho} \right\}$$

Suppose that each of the conditions

$$\mathfrak{D}(a, \Gamma(a))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(a, b)^{\delta^2} \text{ or}$$

$$\mathfrak{D}(b, \Psi(b))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(a, b)^{\delta^2},$$

Implies

$$\mathfrak{D}(\Gamma(a), \Psi(b)) \leq \mathfrak{D}(a, \Gamma(a))^\alpha \cdot \mathfrak{D}(b, \Psi(b))^\beta.$$

$$\mathfrak{D}(a, \Psi(b))^\gamma \cdot \mathfrak{D}(b, \Gamma(a))^\delta \cdot \mathfrak{D}(a, b)^\varrho,$$

for $a, b \in \mathcal{C}$. Then Γ and Ψ get a fixed point v .

Proof. Let $\sigma(\alpha, \beta, \gamma, \delta, \varrho) \in (0, 1)$, since

$$1 - \delta^2\alpha - \delta^2\delta - (\delta^3 \cdot \delta^4)(\gamma \cdot \varrho) < 1 - \delta^3\gamma - \delta^2\delta - \delta^3\varrho$$

and

$$1 - \delta^2\beta - \delta^2\gamma - (\delta^3 \cdot \delta^4)(\delta \cdot \varrho) < 1 - \delta^2\gamma - \delta^3\delta - \delta^3\varrho.$$

Let $u \in \mathcal{C}$. Put $u_0 = u$. For each $i \in \mathbb{N}$, we take

$$u_{2i+1} = \Gamma(u_{2i}) \text{ and } u_{2i} = \Psi(u_{2i-1}). \text{ If } u_0 = u_1, \text{ then}$$

$$u_0 = \Psi(u_0). \text{ Truly, if } u_0 \neq \Psi(u_0) = \Psi(u_1), \text{ then}$$

$$\mathfrak{D}(u_0, \Gamma(u_0))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(u_0, \Gamma(u_0))^{\delta^2}$$

Hence, we get

$$\mathfrak{D}(u_1, u_2) = \mathfrak{D}(\Gamma(u_0), \Psi(u_1)) \leq \mathfrak{D}(u_0, u_1)^\alpha \cdot \mathfrak{D}(u_1, u_2)^\beta.$$

$$\mathfrak{D}(u_0, u_1)^{\delta^\gamma} \cdot \mathfrak{D}(u_1, u_2) \cdot \mathfrak{D}(u_0, u_1)^\varrho$$

It yields

$$\mathfrak{D}(u_1, u_2)^{(1-\beta-\delta^\gamma)} \leq 1,$$

Which shows that $\mathfrak{D}(u_0, \Psi(u_1)) = 1$, a contradiction.

Here Γ and Ψ get u_0 as a fixed point.

Similarly, from the known theorem of (3.2) we get the result that $u_i = u_{i+1}$ for some i , then Γ and Ψ get u_i is the common fixed point.

If $u_0 \neq u_1$ and $u_1 \neq u_2$ we see that

$$\mathfrak{D}(u_0, \Gamma(u_0))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(u_0, \Psi(u_0))^{\delta^2}$$

Since $\sigma(\alpha, \beta, \gamma, \delta, \varrho) < 1 \leq \delta^2$. Then, we get

$$\begin{aligned} \mathfrak{D}(u_1, u_2) &= \mathfrak{D}(\Gamma(u_0), \Psi(u_1)) \\ &\leq \mathfrak{D}(u_0, u_1)^\alpha \cdot \mathfrak{D}(u_1, u_2)^\beta \cdot \mathfrak{D}(u_0, u_2)^\gamma \cdot \mathfrak{D}(u_0, u_1)^\varrho. \end{aligned}$$

From lemma 3.1,

$$\mathfrak{D}(u_1, u_2) \leq \mathfrak{D}(u_0, u_1)^k \text{ where } k = \frac{\alpha + \delta\gamma + \varrho}{1 - \delta\gamma - \beta} \in (0, 1)$$

Similarly, we

$$\mathfrak{D}(u_1, u_2)^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(u_1, u_2)^{\delta^2}.$$

From lemma 3.1,

$$\mathfrak{D}(u_2, u_3) \leq \mathfrak{D}(u_1, u_2)^h \leq \mathfrak{D}(u_0, u_1)^{kh},$$

$$\text{where } h = \frac{\beta + \delta\delta + \varrho}{1 - \delta\delta - \alpha} \in (0, 1)$$

By repeating the above process, we get the same results as we get in theorem (3.2) that is $\{u_i\}$ is sequence and is cauchy then $\lim_{i \rightarrow \infty} u_i = v$ for some $v \in \mathcal{C}$.

To verify that v is a CFP of the mappings Γ and Ψ .

Take $a \neq v$ be number from $\mathcal{C}$. Then $i \in \mathbb{N}$ such that

$$\mathfrak{D}(u_n, v) \leq \frac{1}{2 + \delta} \mathfrak{D}(a, v) \text{ for every } i \in \mathbb{N}.$$

We get

$$\begin{aligned} \mathfrak{D}(a, v) &\leq [\mathfrak{D}(a, u_{2i-1}) \cdot \mathfrak{D}(u_{2i-1}, v)]^S \\ &\leq \left[\mathfrak{D}(a, u_{2i-1}) \cdot \mathfrak{D}(a, v)^{\frac{1}{2+s}} \right]^S \quad \text{for large enough } n, \end{aligned}$$

thus

$$\mathfrak{D}(a, v)^{\frac{2}{2+\delta}} \leq \mathfrak{D}(a, u_{2i-1})^\delta$$

What is more,

$$\begin{aligned} \mathfrak{D}(u_{2i-1}, \Psi(u_{2i-1}))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} &\leq \mathfrak{D}(u_{2n-1}, \Psi(u_{2n-1})) \\ &\leq [\mathfrak{D}(u_{2i-1}, v) \cdot \mathfrak{D}(v, u_{2i})]^S \\ &\leq \mathfrak{D}(a, v)^{\frac{2S}{2+S}} \\ &\leq \mathfrak{D}(a, u_{2i-1})^{\delta^2}. \end{aligned}$$

Hence, we deduce that

$$\mathfrak{D}(\Gamma(a), \Psi(u_{2i-1})) \leq \mathfrak{D}(a, \Gamma(a))^\alpha \cdot \mathfrak{D}(u_{2i-1}, \Psi(u_{2i-1}))^\beta.$$

$$\mathfrak{D}(a, \Psi(u_{2i-1}))^\gamma \cdot \mathfrak{D}(u_{2i-1}, \Gamma(a))^\delta \cdot \mathfrak{D}(a, u_{2i-1})^\varrho.$$

Then

$$\begin{aligned} &\mathfrak{D}(\Gamma(a), v) \\ &\leq [\mathfrak{D}(\Gamma(a), u_{2i}) \cdot \mathfrak{D}(u_{2i}, v)]^S \cdot \mathfrak{D}(a, \Gamma(a))^{S\alpha} \\ &\mathfrak{D}(u_{2i-1}, \Psi(u_{2i-1}))^{S\beta} \cdot [\mathfrak{D}(a, v) \cdot \mathfrak{D}(v, u_{2i})]^{S^2} \\ &[\mathfrak{D}(u_{2i-1}, v) \cdot \mathfrak{D}(\Gamma(a), v)]^{S^2} \cdot \mathfrak{D}(\Gamma(a), v) \cdot \mathfrak{D}(v, u_{2i})^{S^2} \\ &\cdot \mathfrak{D}(u_{2i}, v)^S \\ &\leq \lim_{i \rightarrow \infty} \sup \left\{ \begin{aligned} &\mathfrak{D}(a, \Gamma(a))^{S\alpha} \cdot \mathfrak{D}(u_{2i-1}, \Psi(u_{2i-1}))^{S\beta} \\ &[\mathfrak{D}(a, v) \cdot \mathfrak{D}(a, u_{2i})]^{S^2} \\ &\left\{ [\mathfrak{D}(u_{2i-1}, v) \cdot \mathfrak{D}(\Gamma(a), v)]^{S^2} \cdot [\mathfrak{D}(a, v) \cdot \mathfrak{D}(v, u_{2i-1})]^{S^2} \right\} \\ &\mathfrak{D}(u_{2i}, v)^S \end{aligned} \right\} \\ &\mathfrak{D}(a, \Gamma(a))^{S\alpha} \cdot \mathfrak{D}(a, v)^{(\gamma+\varrho)S^2} \cdot \mathfrak{D}(\Gamma(a), v)^{S^2}. \end{aligned}$$

It follows that

$$\mathfrak{D}(\Gamma(v), v) \leq \mathfrak{D}(\Gamma(v), v)^{\delta\alpha} \cdot \mathfrak{D}(\Gamma(v), v)^{\delta^2\delta},$$

Which implies $\mathfrak{D}(\Gamma(v), v)^{(1-\delta\alpha-\delta^2\delta)} \leq 1$. Notice that

$$1 - \delta\alpha - \delta^2\delta > (\delta^2 - \delta)\alpha \cdot \delta^2\beta \cdot (\delta^3 \cdot \delta^4)(\gamma \cdot \varrho) > 1$$

Hence, we deduce $\mathfrak{D}(\Gamma(v), v) = 1$. By adding

$\mathfrak{D}(\Gamma(a), v)^{-s^3(\gamma+\varrho)}$ on both sides

$$\begin{aligned} &\mathfrak{D}(\Gamma(a), v) \cdot \mathfrak{D}(\Gamma(a), v)^{-\delta^3(\gamma+\varrho)} \\ &\leq \mathfrak{D}(a, \Gamma(a))^{\delta\alpha} \cdot \mathfrak{D}(a, v)^{\delta^2(\gamma+\varrho)} \cdot \mathfrak{D}(\Gamma(a), v)^{-\delta^3(\gamma+\varrho)} \cdot \mathfrak{D}(\Gamma(a), v)^{\delta^2\delta} \\ &\leq \mathfrak{D}(a, \Gamma(a))^{\delta\alpha} \cdot \mathfrak{D}(a, \Gamma(a))^{\delta^3(\gamma+\varrho)} \cdot \mathfrak{D}(\Gamma(a), v)^{\delta^2\delta} \end{aligned}$$

Thus

$$\mathfrak{D}(\Gamma(a), v) \leq \mathfrak{D}(a, \Gamma(a))^{\frac{\delta\alpha \cdot S^3 \gamma \cdot S^3 \varrho}{1 - S^3 \gamma - S^2 \delta - S^3 \varrho}}$$

and

$$\mathfrak{D}(a, \Gamma(a)) \leq [\mathfrak{D}(a, v) \cdot \mathfrak{D}(\Gamma(a), v)]^S$$

$$\leq \mathfrak{D}(a, v) \cdot \mathfrak{D}(a, \Gamma(a))^{\frac{S^2 \alpha \cdot S^4 \gamma \cdot S^4 \varrho}{1 - S^3 \gamma - S^2 \delta - S^3 \varrho}}$$

Therefore, we get

$$\mathfrak{D}(a, \Gamma(a)) \leq \mathfrak{D}(a, v)^{S^2} \frac{1 - \delta^2\alpha - \delta^2\delta - (\delta^3 \cdot \delta^4)(\gamma \cdot \varrho)}{1 - S^2 \delta S^3 - (\gamma \cdot \varrho)}.$$

Similarly, we can also deduce $\mathfrak{D}(\Psi(v), v) = 1$ and

$$\mathfrak{D}(a, \Gamma(a)) \leq \mathfrak{D}(a, v)^{S^2} \frac{1 - \delta^2\beta - \delta^2\gamma - (\delta^3 \cdot \delta^4)(\delta \cdot \varrho)}{1 - S^2 \gamma - S^3(\delta \cdot \varrho)}.$$

Hence Γ and Ψ have v is a common fixed point. We have

$$\mathfrak{D}(v, \Gamma(v))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(a, v)^{\delta^2},$$

$$\mathfrak{D}(a, \Psi(a))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(a, v)^{\delta^2}$$

To prove that Γ and Ψ have a common fixed point

Consider there exists $v, v' \in \mathcal{C}$ so that $\Gamma(v) = \Psi(v) = v$ and $\Gamma(v') = v' = \Psi(v')$.

$$\mathfrak{D}(v, \Gamma(v))^{\sigma(\alpha, \beta, \gamma, \delta, \varrho)} \leq \mathfrak{D}(v, v')^{\delta^2}$$

Then we have

$$\mathfrak{D}(v, v') = \mathfrak{D}(\Gamma(v), \Psi(v'))$$

$$\leq \mathfrak{D}(v, \Psi(v))^{\alpha} \cdot \mathfrak{D}(v', \Psi(v'))^{\beta}.$$

$$\mathfrak{D}(v, \Psi(v'))^{\gamma} \cdot \mathfrak{D}(v', \Gamma(v))^{\delta} \cdot \mathfrak{D}(v, v')^{\varrho}$$

Which yields

$$\mathfrak{D}(v, v')^{(1-\gamma-\delta-\varrho)} \leq 1$$

Since

$$1 - \gamma - \delta - \varrho > \alpha, \beta > 1$$

We get $\mathfrak{D}(v, v') = 1$. Hence the proof.

Theorem 3.4

Let Γ be a continuous self-mapping on a complete b-MMS $(\mathcal{C}, \mathfrak{D})$. Then Γ has a fixed point if $\beta \in (0, 1), \alpha, \gamma, \delta, \varrho \in [0, 1]$ with $\alpha + \beta + \gamma + 2\delta\delta + \varrho < 1$ and a self-map E on $\mathcal{C}$ satisfy the below conditions:

1. $\Psi(\mathcal{C}) \subseteq \Gamma(\mathcal{C})$,
2. Γ and Ψ commute under composition (i.e., $\Psi(\Gamma(a)) = \Gamma(\Psi(a))$ for all $a \in \mathcal{C}$),
3. $\mathfrak{D} \begin{cases} \Psi(a), \Psi(b) \leq \mathfrak{D}(\Gamma(a), \Psi(a))^{\alpha} \cdot \\ \mathfrak{D}(\Gamma(b), \Psi(b))^{\beta} \cdot \mathfrak{D}(\Gamma(a), \Psi(b))^{\gamma} \end{cases}$

$$\mathfrak{D}(\Gamma(b), \Psi(a))^{\delta} \cdot \mathfrak{D}(\Gamma(a), \Psi(b))^{\varrho},$$

for all $a, b \in \mathcal{C}$.

If the above are conditions satisfy then Γ and Ψ have a unique common fixed point.

Proof. If Γ has a fixed point $v \in \mathcal{C}$, then we take $\Psi: \mathcal{C} \rightarrow \mathcal{C}$ by $\Gamma(a) = v$ for all $a \in \mathcal{C}$.

It shows that $\Psi(\mathcal{C}) \subseteq \Gamma(\mathcal{C})$. Then for any $a \in \mathcal{C}, \Psi(\Gamma(a)) = v$ and $\Gamma(\Psi(a)) = \Gamma(v) = v$, which implies that $\Psi(\Gamma(a)) = \Gamma(\Psi(a))$ for all $a \in \mathcal{C}$.

For any $\alpha, \beta, \gamma, \delta, \varrho \in [0, 1]$ and $a, v \in \mathcal{C}$,

$$\mathfrak{D}(\Psi(a), \Psi(b)) = \mathfrak{D}(v, v) = 1$$

$$\leq \mathfrak{D}(\Gamma(a), v)^{\alpha} \cdot \mathfrak{D}(\Gamma(b), v)^{\beta} \cdot \mathfrak{D}(\Gamma(a), v)^{\gamma}.$$

$$\mathfrak{D}(\Gamma(b), v)^{\delta} \cdot \mathfrak{D}(\Gamma(a), \Gamma(b))^{\varrho}.$$

However, we assume that Γ has a fixed point, if Γ and Ψ have a unique fixed point by satisfy the conditions of (1) through (3) are true. Let $a_0 \in \mathcal{C}$ in order to view this. Condition (1) shows that $a_1 \in \mathcal{C}$ in such a way that $\Gamma(a_1) = \Psi(a_0)$. Continuing in this procedure again and again, we could get $\{a_i\} \in \mathcal{C}$ then

$$\Gamma(a_i) = \Psi(a_{i-1}) \text{ for } i \in \mathbb{N}.$$

$$\mathfrak{D}(\Gamma(a_{i+1}), \Gamma(a_i)) = \mathfrak{D}(E(a_i), E(a_{i-1}))$$

$$\leq \mathfrak{D}(\Gamma(a_n), \Gamma(a_{n+1}))^{\alpha} \cdot \mathfrak{D}(\Gamma(a_{n-1}), \Gamma(a_n))^{\beta}.$$

$$[\mathfrak{D}(\Gamma(a_{n-1}), \Gamma(a_n)) \cdot \mathfrak{D}(\Gamma(a_n), \Gamma(a_{n+1}))]^{\delta\delta\varrho}$$

$$\cdot \mathfrak{D}(\Gamma(a_n), \Gamma(a_{n-1})).$$

Then

$$\mathfrak{D}(\Gamma(a_{i+1}), \Gamma(a_i)) \leq \mathfrak{D}(\Gamma(a_i), \Gamma(a_{i-1}))^{\frac{\beta + \delta\delta + \varrho}{1 - \alpha - s\delta}}.$$

Because $\frac{\beta + s\delta + \varrho}{1 - \alpha - s\delta} < 1$, we find that $\{\Gamma(a_i)\}$ is a

Cauchy sequence. By the completeness of the space, we conclude that $\lim_{i \rightarrow \infty} \Gamma(a_i) = v$ for some $v \in \mathcal{C}$. From

$$\Gamma(a_i) = \Gamma(a_{i-1}), \text{ Consequently } \lim_{n \rightarrow \infty} \Psi(a_i) = v. \text{ By}$$

condition (2), we fine

$$\Gamma(v) = \lim_{i \rightarrow \infty} \Gamma(\Psi(a_i)) = \lim_{i \rightarrow \infty} \Psi(\Gamma(a_i)) = \Gamma(v),$$

Which yields

$$\Gamma(\Gamma(v)) = \Gamma(\Psi(v)) = \Psi(\Gamma(v)) = \Psi(\Psi(v)).$$

By condition (3),

$$\mathfrak{D}(\Psi(v), \Psi(E(v)))$$

$$\leq \mathfrak{D}(\Gamma(v), \Psi(v))^{\alpha} \cdot \mathfrak{D}(\Gamma(\Psi(v)), \Psi(\Psi(v)))^{\beta} \cdot \mathfrak{D}(\Gamma(v), \Psi(\Psi(v)))^{\gamma}$$

$$\mathfrak{D}(\Gamma(\Psi(v)), \Psi(v))^{\delta} \cdot \mathfrak{D}(\Gamma(v), \Gamma(\Psi(v)))^{\varrho}$$

$$= \mathfrak{D}(\Gamma(v), \Gamma(\Gamma(v)))^{\gamma} \cdot \mathfrak{D}(\Gamma(v), \Gamma(\Gamma(v)))^{\delta} \cdot \mathfrak{D}(\Gamma(v), \Gamma(\Gamma(v)))^{\varrho}$$

Which implies $\mathfrak{D}(\Gamma(v), \Gamma(\Gamma(v)))^{(1-\gamma-\delta-\varrho)} \leq 1$. Since $1 - \gamma - \delta - \varrho > 1$, for uniqueness, if there exists $v, p \in \Gamma$ such that $\Gamma(v) = v = \Psi(v)$ and $\Gamma(p) = p = \Psi(p)$, then by condition (3), it follows that $\Gamma(z)$ is a common fixed point of Γ and Ψ .

$$\begin{aligned}
\mathfrak{D}(v, p) &= \mathfrak{D}(\Psi(v), \Psi(p)) \\
&\leq \mathfrak{D}(\Gamma(v), \Psi(v))^\alpha \cdot \mathfrak{D}(\Gamma(p), \Psi(p))^\beta \cdot \mathfrak{D}(\Gamma(v), \Psi(p))^\gamma \\
&\quad \cdot \mathfrak{D}(\Gamma(p), \Psi(v))^\delta \cdot \mathfrak{D}(\Gamma(v), \Gamma(p))^\varrho \\
&= \mathfrak{D}(v, p)^\gamma \cdot \mathfrak{D}(v, p)^\delta \cdot \mathfrak{D}(v, p)^\varrho.
\end{aligned}$$

Clearly, we have $\mathfrak{D}(v, p)^{(1-\gamma-\delta-\varrho)} \leq 1$. Further $1-\gamma-\delta-\varrho > 1$, we get $\mathfrak{D}(v, p) = 1$, which suggests $v = p$.

Conclusion

In this research article, we explored fixed point theorems for two self-mappings and generalized Jungck's fixed point theorem using $\mathbb{F}$ -contractions with weakened conditions. We provided illustrative examples to demonstrate our findings, focusing on non-decreasing functions. This work enables the solution of differential and integral equations. However, an open problem remains: investigating the applicability of the same F-contraction to decreasing functions in fixed point theory.

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Conflicts of interest

The authors declare no conflict of interest.

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