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ORIGINAL RESEARCH PAPER

A pair of fractional power of generalized hankel-clifford type transformations and their characteristics

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Abstract

This research introduces a pair of the fractional powers of generalized Hankel-Clifford transforms and their associated differential operators. It also explores mathematical properties and their behavior in Zemanian-type spaces.

Keywords: Fractional Hankel-Clifford transforms, Differential operators, Zemanian type spaces.

2020 Mathematics subject classification: 46F12, 44A05, 46F05, 46E10.

Introduction

Studying integral transformations and their related differential operators has long been a key component of mathematical analysis. It provides effective techniques for solving problems in differential equations, signal processing, and various fields of science and engineering (Graf, 2010; Pathak, 2017). The Hankel–Clifford transform is one such transformation that has gained attention for its utility in boundary value problems and its connections to higher-dimensional spaces. Over time, this transformation has been generalized and extensively studied by researchers, including (Prasad *et al.*, 2012), who work on the establishment of the pseudo-differential operator as a continuous linear map on Zemanian-type spaces with integral representation and norm inequality using Hankel–Clifford transformations. (Prasad & Kumar, 2014), whose work on fractional powers of Hankel–Clifford transformations and pseudo-differential operators significantly contributed to this area.

Further developments by (Gorty, 2014) on the computational aspects of discrete generalized Hankel-Clifford transform, followed by his study on continuous fractional forms, laid the groundwork (Gorty, 2015). (Prasad & Kumar, 2016) have investigated compositions of pseudo-differential operators associated with fractional Hankel–Clifford transformations, expanding their application in functional analysis. Additionally, the theoretical and practical aspects of fractional Hankel–Clifford transforms within Sobolev-type spaces have been explored by (Kumar & Pradhan, 2022), with recent research by (Waphare & Shaikh, 2023) examining fractionalization techniques for these transforms. Our research is motivated by all of these results.

Our study aims to introduce and characterize a pair of fractional power generalized Hankel-Clifford transforms. We will explain their mathematical properties and behavior in Zemanian-type spaces defined in this paper.

Let us begin with the traditional definition of a pair of Hankel-Clifford transforms of order was initially introduced by (Betancor, J.J., 1989) and (Pérez & Robayna 1991) and is defined as

$$\left(h_{1,\mu}\varphi\right)(y) = y^{\mu}\int_0^{\infty} C_{\mu}(xy)\varphi(x)\,dx, \#(1.1)$$

$$\left(h_{2,\mu}\varphi\right)(y) = \int_0^{\infty} x^{\mu} C_{\mu}(xy)\varphi(x)\,dx, \#(1.2)$$

where $C_{\mu}(xy) = \ddot{u}xy^{-\mu/2} J_{\mu}\left(\sqrt{xy}\right)$ is the first kind of Bessel-Clifford function of order μ.

Later, (Malgonde, S.P. & Bandewar, S.R., 2000) introduced generalized Hankel-Clifford transformations of order

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$\left(\alpha \aleph \beta\right)$ ü , which has two real parameters α and β, and it is further investigated by (Pansare & Waphare, 2016) and is defined by

$$\hat{\varphi}(y) = \left(h_{1,\alpha,\beta}\varphi\right)(y) = y^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta}(xy)\varphi(x)\,dx, \#(1.3)$$

$$\tilde{\psi}(y) = \left(h_{2,\alpha,\beta}\psi\right)(y) = \int_0^\infty x^{\alpha-\beta}C_{\alpha,\beta}(xy)\psi(x)\,dx, \#(1.4)$$

where $\ddot{u}_{\alpha,\beta}(\quad)$ is the first kind of generalized Bessel-Clifford function of order $\alpha-\beta$ defined by

$$C_{\alpha,\beta}(xy) = (xy)^{\frac{-\alpha+\beta}{2}} J_{\alpha-\beta}\left(2\sqrt{xy}\right).$$

Here, we use these generalized Hankel-Clifford transformations to introduce the fractional power of generalized Hankel-Clifford transformations of order $a-b$, which depends on α, β

$$\left(h^\theta_{1,\alpha,\beta,a,b}\varphi\right)(y) = \hat{\varphi}^\theta_{1,\alpha,\beta,a,b}(y) = \int_0^\infty K_1^\theta(x,y)\varphi(x)\,dx, \#(1.5)$$

where,

$$K_1^\theta(x,y) = \begin{cases} \gamma_{a,b,\theta}\, y^{\alpha-\beta} C_{\alpha,\beta,a,b}(\mathrm{xycsc}\theta)\, e^{\left(\frac{i}{2}(x^2+y^2)\ddot{u}\ \theta\right)} & \theta \neq n\pi, \\ y^{\alpha-\beta}C_{\alpha,\beta,a,b}(xy) & \theta = \frac{\pi}{2}, \\ \delta(x-y) & \theta = n\pi. \end{cases} \#(1.6)$$

And .

$$C_{\alpha,\beta,a,b}(xy) = (xy)^{(-\alpha+\beta)/2} J_{a-b}\left(2\sqrt{xy}\right);\ \gamma_{a,b,\theta}$$

$$= \exp\left[i(a-b+1)(\pi/2-\theta)\right]$$

Similarly,

$$\left(h^\theta_{2,\alpha,\beta,a,b}\psi\right)(y) = \tilde{\psi}^\theta_{2,\alpha,\beta,a,b}(y) = \int_0^\infty K_2^\theta(x,y)\psi(x)\,dx, \#(1.7)$$

where,

$$K_2^\theta(x,y) = \begin{cases} \gamma_{a,b,\theta}\, x^{\alpha-\beta} C_{\alpha,\beta,a,b}(\mathrm{xycsc}\theta)\, e^{\left(\frac{i}{2}(x^2+y^2)\ddot{u}\ \theta\right)} & \theta \neq n\pi, \\ x^{\alpha-\beta}C_{\alpha,\beta,a,b}(xy) & \theta = \frac{\pi}{2}, \\ \delta(x-y) & \theta = n\pi. \end{cases} \#(1.8)$$

Note that $\left(h^\theta_{2,\alpha,\beta,a,b}\psi\right)(y) = y^{\alpha-\beta}\left(h^\theta_{1,\alpha,\beta,a,b}\varphi\right)(y)$.

The inverse is defined as,

$$\varphi = \left(\left(h^\theta_{1,\alpha,\beta,a,b}\right)^{-1}\hat{\varphi}^\theta_{1,\alpha,\beta,a,b}\right)(x) = \int_0^\infty \bar{K}_1^\theta(y,x)\hat{\varphi}^\theta{}_{1,\alpha,\beta,a,b}(y)\,dy, \#(1.9)$$

$$\psi = \left(\left(h^\theta_{2,\alpha,\beta,a,b}\right)^{-1}\tilde{\psi}^\theta_{2,\alpha,\beta,a,b}\right)(x) = \int_0^\infty \bar{K}_2^\theta(x,y)\tilde{\psi}^\theta_{2,\alpha,\beta,a,b}(y)\,dy, \#(1.10)$$

where, $\bar{K}_1^\theta$ and $\bar{K}_2^\theta$ are conjugates of K_1^θ and K_2^θ respectively.

Parsevals's Identity

$$\int_0^\infty x^{\beta-\alpha}\varphi(x)\bar{\psi}(x)\,dx = \int_0^\infty y^{\beta-\alpha}\left(h^\theta_{1,\alpha,\beta,a,b}\varphi\right)(y)\left(\overline{h^\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y)\,dx,$$

is Parseval's identity for (1.5) and for (1.7) we have,

$$\int_0^\infty x^{\alpha-\beta}\varphi(x)\bar{\psi}(x)\,dx = \int_0^\infty y^{\alpha-\beta}\left(h^\theta_{2,\alpha,\beta,a,b}\varphi\right)(y)\left(\bar{h}_{2,\alpha,\beta,a,b}\varphi\right)(y)\,dx.$$

We can also express mixed Parseval's identity using Fubuni's theorem as

$$\int_0^\infty \varphi(x)\bar{\psi}(x)\,dx = \int_0^\infty \left(h^\theta_{1,\alpha,\beta,a,b}\varphi\right)(y)\left(\bar{h}^\theta{}_{2,\alpha,\beta,a,b}\varphi\right)(y)\,dx. \#(1.11)$$

Also,

$$D_x^r C_{\alpha\ddot{u}\beta}(x) = (-1)^r C_{\alpha\ \beta+r}(x);\ D_x^r\left[x^{\alpha-\beta+r}C_{\alpha\ \beta+r}(x)\right]$$

$$= x^{\alpha-\beta}C_{\alpha,\beta+r}(x)\ \forall r \in \mathbb{N}_0. \#(1.12)$$

Preliminary

Let us introduce some differential operators corresponding to the fractional power of generalized Hankel-Clifford transforms followed by (Torre, 2008) as

$$R_{1,\alpha,\beta,a,b,\theta} = e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{\alpha-\beta+a-b+1}{2}} D_x x^{\frac{-\alpha+\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}, \#(1.13)$$

$$S_{1,\alpha,\beta,a,b,\theta} = e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{\alpha-\beta-a+b}{2}} D_x x^{\frac{-\alpha+\beta+a-b+1}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}. \#(1.14)$$

Also, the Kepinski-type differential operator is defined as

$$\Delta_{1,\alpha,\beta,a,b,\theta} = S_{1,\alpha,\beta,a,b,\theta}R_{1,\alpha,\beta,a,b,\theta}$$

$$= e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{\alpha-a-\beta+b}{2}} D_x x^{a-b+1} D_x x^{\frac{-\alpha-a+\beta+b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)} \quad \#(1.15)$$

$$= xD_x^2 + \left[(1-\alpha+\beta) - 2ix^2\cot\theta\right]D_x - \left[(2-\alpha+\beta)\mathrm{ixcot}\theta + x^3\cot^2\theta\right.$$

$$\left. + \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\right]. \quad \#(1.16)$$

Similarly,

$$R_{2,\alpha,\beta,a,b,\theta} = -e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{-\alpha+\beta-a+b}{2}} D_x x^{\frac{\alpha-\beta+a-b+1}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}, \#(1.17)$$

$$S_{2,\alpha,\beta,a,b,\theta} = -e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{-\alpha+\beta+a-b+1}{2}} D_x x^{\frac{\alpha-\beta-a+b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}. \quad \#(1.18)$$

$$\Delta_{2,\alpha,\beta,a,b,\theta} = R_{2,\alpha,\beta,a,b,\theta}S_{2,\alpha,\beta,a,b,\theta}$$

$$= e^{\left(\frac{i}{2}x^2\cot\theta\right)} x^{\frac{-\alpha-a+\beta+b}{2}} D_x x^{a-b+1} D_x x^{\frac{\alpha-a-\beta+b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)} \quad \#(1.19)$$

$$= xD_x^2 + \left[(1+\alpha-\beta) - 2ix^2\cot\theta\right]D_x - \left[(2+\alpha-\beta)\,\mathrm{ixcot}\theta + x^3\cot^2\theta\right.$$

$$\left. + \frac{a^2+b^2-\alpha^2-\beta^2+2(\alpha\beta-ab)}{4x}\right]. \qquad (1.20)$$

Note that if $\alpha=\beta=0, a=b=0, \theta=\pi/2$ in $\Delta_{1,\alpha,\beta,a,b,\theta}$ and $\Delta_{2,\alpha,\beta,a,b,\theta}$ then, in both cases we get

$$\Delta_{1,0,0,0,0,\frac{\pi}{2}} = xD_x^2 + D_x; \Delta_{2,0,0,0,0,\pi/2} = xD_x^2 + D_x.$$

Thus, $\Delta_1 = \Delta_2$

And for $\alpha=\beta, a=b, \theta=\pi/2$ we have,

$$\Delta_{1,\alpha,\alpha,a,a,\frac{\pi}{2}} = xD_x^2 + D_x; \Delta_{2,\alpha,\alpha,a,a,\pi/2} = xD_x^2 + D_x.$$

Thus, $\Delta_1 = \Delta_2$

The spaces $\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$ and $\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I)$

In this section, we present the Zemanian-type spaces motivated by (Zemanian, A. H. 1962), the spaces $\mathcal{H}^{\theta}_{1,\alpha,\beta,\ a,b}(I)$, $\mathcal{H}^{\theta}_{2,\alpha,\beta,\ a,b}(I)$, and their duals.

$\mathcal{H}^{\theta}_{1,\alpha,\beta,\ a,b}(I)$ is space consisting of all the complex-valued smooth function φ defined on I such that, for $\forall q,k \in \mathbb{N}_0$, it satisfies

$$\rho^{1,\alpha,\beta,a,b}_{q,k,\theta}(\varphi) \quad = \sup_{x\in I}\left|x^q D_x^k x^{-\frac{\alpha+a-\beta-b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}\varphi(x)\right|$$

$$= \sup_{x\in I}\left|x^q D_x^k x^{-\frac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2 \text{ü}\ \ \theta\right)}\varphi(x)\right| < \infty. \qquad (1.21)$$

Also, the family of seminors $\left\{\rho^{1,\alpha,\beta,\ a,b}_{q,k,\theta}\right\}_{q,k\in\mathbb{N}_0}$ generate topology over $\mathcal{H}^{\theta}_{1,\alpha,\beta,\ a,b}(I)$.

$\mathcal{H}^{\theta}_{2,\alpha,\beta,\ a,b}(I)$ is space consisting of all the complex-valued smooth function φ defined on I such that, for $\forall q,k \in \mathbb{N}_0$, it satisfies

$$\rho^{2,\alpha,\beta,\ a,b}_{q,k,\theta}(\varphi) \quad = \sup_{x\in I}\left|x^q D_x^k x^{\frac{\alpha-\mathrm{a}-\beta+b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}\varphi(x)\right|$$

$$= \sup_{x\in I}\left|x^q D_x^k x^{\frac{\alpha-\mathrm{a}-\beta+b}{2}} e^{\left(\frac{i}{2}x^2 \text{ü}\ \ \theta\right)}\varphi(x)\right| < \infty. \qquad (1.22)$$

Also, the family of seminors $\left\{\rho^{2,\alpha,\beta,\ a,b}_{q,k,\theta}\right\}_{q,k\in\mathbb{N}_0}$ generate topology over $\mathcal{H}^{\theta}_{1,\alpha,\beta,\ a,b}(I)$.

Now, consider a function $\psi(x)$, which is locally integrable and of slow growth as $x\to\infty$ such that $\text{ü}^{\frac{\alpha \aleph a\ \ \beta\ \ b}{2}} \left(\frac{i}{2}x^2\text{ü}\ \ \theta\right)\psi(\ )$ is absolutely integrable on I. Then ψ generates regular generalized functions in the dual of $\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}\mathrm{viz}, \psi \in \left(\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}\right)^{'}(I)$ by

$$\langle\psi,\varphi\rangle = \int_0^\infty \psi(x)\varphi(x)dx, \ \ \varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$$

$$= \int_0^1 x^{\frac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2\cot\theta\right)}\psi(x) x^{-\frac{\alpha+a-\beta-b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}\psi(x)dx + \int_1^\infty x^{-m+\frac{\alpha+a-\beta-b}{2}}$$

$$\times e^{\left(\frac{i}{2}x^2\cot\theta\right)}\psi(x) x^{m-\frac{\alpha+a-\beta-b}{2}} e^{\left(-\frac{i}{2}x^2\cot\theta\right)}\psi(x)dx.$$

As $\psi(x)$ is of slow growth, let us choose m large enough so that $\text{ü}^{\frac{\alpha\aleph a\ \ \beta\ \ b}{2}-m} \left(\frac{i}{2}x^2\text{ü}\ \ \theta\right)\psi(\ )$ is absolutely convergent on I. Then,

$$\left|\langle\psi,\varphi\rangle\right| \le \rho^{1,\alpha,\beta,a,b}_{0,0,\theta}(\varphi)\int_0^1\left|x^{\frac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2\cot\theta\right)}\psi(x)\right|dx$$

$$+\rho^{1,\alpha,\beta,a,b}_{m,0,\theta}(\varphi)\int_1^\infty\left|x^{-m+\frac{\alpha+a-\beta-b}{2}} e^{\left(\frac{i}{2}x^2\cot\theta\right)}\psi(x)\right|dx.$$

Hence, $\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I) \subset \left(\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}\right)^{'}(I)$.

Following a similar way by considering a locally integrable function ψ of slow growth as $x\to\infty$ and $\text{ü}^{\frac{-\alpha+\mathrm{a}+\beta-b}{2}} \left(\frac{i}{2}x^2\text{ü}\ \ \theta\right)\psi(\ )$ is absolutely integrable on I. Then ψ generates regular generalized function ψ in $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)^{'}(I)$ by

$$<\psi,\varphi> = \int_0^\infty \psi(x)\varphi(x)dx, \qquad \varphi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I).(1.23)$$

Hence, we have $\mathcal{H}^{\theta}_{1,\alpha,\beta,\ a,b}(I) \subset \left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)^{'}(I)$. Also, observe that $\left(\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}\right)^{'}(I)$ and $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)^{'}(I)$ are complete.

Characteristics of fractional powers of generalized Hankel-Clifford transformation

Here, we state some properties of fractional powers of generalized Hankel-Clifford transformation and its differential operators.

1. If $\alpha,\beta\in\mathbb{R}, a\geq b$ and $q\in\mathbb{N}_0$ then

$$R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\varphi(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+q}{2}}D_y^q y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}x^2\cot\theta}\varphi(y),\#(2.1)$$

where $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$.

Proof. Given $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$, then from (1.13), we have

$$R_{1,\alpha,\beta,a,b,\theta}\varphi(y)=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+1}{2}}D_y y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}\varphi(y)$$

$$R_{1,\alpha,\beta,a+1,b,\theta}R_{1,\alpha,\beta,a,b,\theta}\varphi(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a+1-\beta-b+1}{2}}D_y y^{\frac{-\alpha-a-1+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}R_{1,\alpha,\beta,a,b,\theta}\varphi(y)$$

$$=\quad e^{\frac{i}{2}y^{\text{ü}}\cot\theta}y^{\frac{\alpha+a+\text{ü}-\beta-b+}{2}}D_y y^{\frac{-\alpha-a-\ +\beta+b}{2}}e^{-\frac{i}{2}y\ \cot\theta}\left\{e^{\frac{i}{2}y\ \cot\theta}\right.$$

$$\left.\times y^{\frac{\alpha+a-\beta-b+1}{2}}D_y y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}\right\}\varphi(y)$$

$$=\qquad e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+2}{2}}D_y^2 y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}\varphi(y).$$

Continuing in this manner, we get the desired outcome.

$$R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\varphi(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+q}{2}}D_y^q y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}x^2\cot\theta}\varphi(y).$$

2. If $\alpha,\beta\in\mathbb{R}, a\geq b$ and $q,k\in\mathbb{N}_0$ then

$$R_{1,\alpha,\beta,a+k+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a+k,b,\theta}(y\csc\theta)^{\frac{k}{2}}\varphi(y)$$

$$=(y\csc\theta)^{\frac{k}{2}}R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\varphi(y),(2.2)$$

where $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$.

Proof. From (1.13), we have,

$$R_{1,\alpha,\beta,a+k,b,\theta}(y\csc\theta)^{k/2}\varphi(y)\quad=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a+k-\beta-b+1}{2}}D_y y^{\frac{-\alpha-a-k+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}(y\csc\theta)^{k/2}\varphi(y)$$

$$=(y\csc\theta)^{k/2}e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+1}{2}}D_y y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}\varphi(y)$$

$$=(y\csc\theta)^{\frac{k}{2}}R_{1,\alpha,\beta,a,b,\theta}\varphi(y).$$

Next, consider

$$R_{1,\alpha,\beta,a+k+1,b,\theta}R_{1,\alpha,\beta,a+k,b,\theta}(y\csc\theta)^{k/2}\varphi(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a+k+\text{ü}-\beta-b+}{2}}D_y y^{\frac{-\alpha-a-k-\ +\beta+b}{2}}e^{-\frac{i}{2}y^2\cot\theta}(y\csc\theta)^{k/2}R_{1,\alpha,\beta,a,b,\theta}\varphi(y)$$

$$=(y\csc\theta)^{\frac{k}{2}}R_{1,\alpha,\beta,a+1,b,\theta}R_{1,\alpha,\beta,a,b,\theta}\varphi(y).$$

Continuing in the same way until we get

$$R_{1,\alpha,\beta,a+k+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a+k,b,\theta}(y\csc\theta)^{\frac{k}{2}}\varphi(y)$$

$$=(y\csc\theta)^{\frac{k}{2}}R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\varphi(y).$$

Lemma 2.1. If $\alpha,\beta\in\mathbb{R}, a\geq b$ and $q\in\mathbb{N}_0$ then

$$R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=(-1)^q e^{-iq\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((\mathrm{x}\csc\theta)^{\frac{q}{2}}\varphi\right)(y),$$

where $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$.

Proof. From (2.1) and (1.5).

$$R_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R_{1,\alpha,\beta,a+k,b,\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+q}{2}}D_y^q y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}x^2\cot\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+\mathrm{a}-\beta-b+q}{2}}D_y^q y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}x^2\cot\theta}y^{\alpha-\beta}\gamma_{a,b,\theta}\int_0^\infty C_{\alpha,\beta,a,b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\cot\theta}\varphi(x)dx$$

$$=\gamma_{a,b,\theta}e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha+a-\beta-b+q}{2}}\int_0^\infty D_y^q\left\{y^{\frac{\alpha-a-\beta+b}{2}}(xy\csc\theta)^{-\frac{\alpha-a-\beta+b}{2}}C_{a,b}(xy\csc\theta)\right\}e^{\frac{i}{2}x^2\cot\theta}\varphi(x)dx$$

$$=(-1)^q\gamma_{a\text{ü}b\ \theta}y^{\frac{\alpha+a-\beta-b+q}{\text{ü}}}\int_0^\infty(x\csc\theta)^{-\frac{\alpha-a-\beta+b}{}+q}C_{a+q\ b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\text{ü}\ \ \theta}\varphi(x)dx$$

$$=(-1)^q\gamma_{a,b,\theta}y^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a+q,b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\text{ü}\ \ \theta}(x\csc\theta)^{q/2}\varphi(x)dx$$

$$=(-1)^q e^{-iq\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((\mathrm{x}\csc\theta)^{\frac{q}{2}}\varphi\right)(y)$$

Lemma 2.2. If and then

$$h^{\theta}_{1,\alpha,\beta,a+q+k,b}\left((x\csc\theta)^{q/2}R^*_{1,\alpha,\beta,a+k-1,b,\theta}\cdots R^*_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$$

$$=(-1)^k e^{ik\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((\mathrm{x}\csc\theta)^{\frac{q}{2}}\varphi\right)(y),$$

where $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$ and

$R^*_{1,\alpha,\beta,a,b,\theta}=e^{-\frac{i}{2}x^2\cot\theta}x^{\frac{\alpha+a-\beta-b+1}{2}}D_x x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^2\cot\theta}$ is adjoint of $R_{2,\alpha,\beta,a,b,\theta}$.

Proof. First, Consider the following and use integration by part,

$$h^{\theta}_{1,\alpha,\beta,a+q+1,b}\left((x\csc\theta)^{q\text{ü}}\ R_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$$

$$=\gamma_{a+q+1,b,\theta}y^{\alpha-\beta}\int_0^{\infty}C_{\alpha,\beta,a+q+1,b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\cot\theta}(x\csc\theta)^{q/2}R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi(x)\,dx$$

$$=\gamma_{a+q+1,b,\theta}y^{\alpha-\beta}\int_0^{\infty}(xy\csc\theta)^{\frac{-\alpha+a+q+1+\beta-b}{2}}C_{a+q+1,b}(xy\csc\theta)(x\csc\theta)^{q/2}e^{\frac{i}{2}(x^2+y^2)\cot\theta}$$
$$\times e^{-\frac{i}{2}x^2\cot\theta}x^{\frac{\alpha+a-\beta-b+1}{2}}D_x x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^2\cot\theta}\varphi(x)\,dx$$
$$=\gamma_{a+q+1,b,\theta}y^{\alpha-\beta}(y\csc\theta)^{-\frac{\alpha+a+1-\beta-b}{2}}y^{-q/2}\int_0^{\infty}(xy\csc\theta)^{a+q+1-b}C_{a+q+1,b}(xy\csc\theta)$$
$$\times D_x x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}(x^2+y^2)\cot\theta}\varphi(x)\,dx$$
$$=\gamma_{a+q+1,b,\theta}y^{\alpha-\beta}(y\csc\theta)^{-\frac{\alpha+a+1-\beta-b}{2}}y^{-q/2}\int_0^{\infty}(y\csc\theta)(xy\csc\theta)^{a+q-b}C_{a+q,b}(xy\csc\theta)$$
$$\times x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}(x^2+y^2)\cot\theta}\varphi(x)\,dx$$
$$=\gamma_{a+q+1,b,\theta}y^{\alpha-\beta}(y\csc\theta)^{1/2}\int_0^{\infty}C_{\alpha,\beta,a+q,b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\cot\theta}(x\csc\theta)^{q/2}\varphi(x)\,dx$$
$$=(-1)e^{i(\pi/2-\theta)}(y\csc\theta)^{1/2}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((x\csc\theta)^{q/2}\varphi\right)(y).$$

Next, by a similar process, we get

$$h^{\theta}_{1,\alpha,\beta,a+q+2,b}\left((x\csc\theta)^{\frac{q}{2}}R^{*}_{1,\alpha,\beta,a+1,b,\theta}R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$$

$$=(-1)^2e^{2i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)h^{\theta}_{1,\alpha,\beta,a+q,b}\left((x\csc\theta)^{\frac{q}{2}}\varphi\right)(y).$$

Continuing similarly, we obtain

$$h^{\theta}_{1,\alpha,\beta,a+q+k,b}\left((x\csc\theta)^{\frac{q}{2}}R^{*}_{1,\alpha,\beta,a+k-1,b,\theta}\cdots R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$$

$$=(-1)^k e^{ik\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((x\csc\theta)^{\frac{q}{2}}\varphi\right)(y).$$

Hence proof is complete.

The following two theorems will be the main findings of this section.

Theorem 2.3. The fractional power of the first generalized Hankel-Clifford type transformation $h^{\theta}_{1,\alpha,\beta,\,a,b}$ is an isomorphism from $\mathcal{H}^{\theta}_{1,\alpha,\beta,\,a,b}(I)$ to itself. Furthermore, $\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)^{-1}$ is a linear and continuous map from $\mathcal{H}^{\theta}_{1,\alpha,\beta,\,a,b}(I)$ to itself.

Proof. Consider $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$ then by Lemma 2.1 we have,

$$R_{1,\alpha,\beta,a+k-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=(-1)^q e^{-iq\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+q,b}\left((x\csc\theta)^{\frac{q}{2}}\varphi\right)(y).$$

Using Lemma 2.2, the above equation becomes

$$R_{1,\alpha,\beta,a+k-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=\frac{(-1)^k e^{-ik\left(\frac{\pi}{2}-\theta\right)}}{(-1)^q e^{iq\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{q}{2}}}h^{\theta}_{1,\alpha,\beta,a+q+k,b}\left((x\csc\theta)^{\frac{k}{2}}R^{*}_{1,\alpha,\beta,a+q-1,b,\theta}\cdots R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y).$$

Hence,

$$(-1)^{k+q}(y\csc\theta)^{q/2}R_{1,\alpha,\beta,a+k-1,b,\theta}\cdots R_{1,\alpha,\beta,a,b,\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)$$

$$=e^{-i(k+q)(\pi/2-\theta)}\gamma_{a+k+q,b,\theta}y^{\alpha-\beta}(y\csc\theta)^{\text{ü}}\int_0^{\infty}C_{\alpha,\beta,a+k+q,b}(xy\csc\theta)e^{\frac{i}{2}(x^2+y^2)\text{ü}\ \theta}$$
$$\times(x\csc\theta)^{k/2}e^{-\frac{i}{2}x^2\cot\theta}x^{\frac{\alpha+a+q-\beta-b}{2}}D_x^q x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^2\cot\theta}\varphi(x)\,dx$$
$$=\gamma_{a,b,\theta}y^{\alpha-\beta}\int_0^{\infty}(xy\csc\theta)^{\frac{-\alpha+a+k+q+\beta-b}{2}}C_{a+k+q,b}(xy\csc\theta)(x\csc\theta)^{k/2}x^{\frac{\alpha+a+q-\beta-b}{2}}$$
$$\times D_x^q x^{\frac{-\alpha-\text{ü}+\beta+}{2}}e^{\frac{}{2}(x^2+y^2)\text{ü}\ \theta}\varphi(x)\,dx$$

Therefore,

Now from above and for $n\geq a+k-b$ we have $x^{\text{ü}+\ -}<\left(1+x^2\right)^n$, which gives,

$$\left|(-1)^{k+q}y^qD_y^k y^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}y^2\cot\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\right|$$

$$\leq E_{1,\alpha,\beta,a,b,\theta,k,q}\sum_{r=0}^{n+1}\binom{n+1}{r}\int_0^{\infty}\left|x^rD_x^q x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^2\cot\theta}\varphi(x)\right|\frac{dx}{1+x^2}$$

$$\leq\frac{\pi}{2}E_{1,\alpha,\beta,a,b,\theta,k,q}\sum_{r=0}^{n+1}\binom{n+1}{r}\sup_{x\in I}\left|x^rD_x^q x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^2\cot\theta}\varphi(x)\right|$$

$$\leq\frac{\pi}{2}E_{1,\alpha,\beta,a,b,\theta,k,q}\sum_{r=0}^{n+1}\binom{n+1}{r}\rho^{1,\alpha,\beta,a,b}_{q,k,\theta}(\varphi).\ \#(2.4)$$

Therefore $h^{\theta}_{1,\alpha,\beta,a,b}\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$.

Also from (1.5) and (1.9),

$$\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{-1}h^{\theta}_{1,\alpha,\beta,a,b}\varphi=\varphi=h^{\theta}_{1,\alpha,\beta,a,b}\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{-1}\varphi,\#(2.5)$$

which clearly shows that $h^{\theta}_{1,\alpha,\beta,\,a,b}$ is a linear injective map from $\mathcal{H}^{\theta}_{1,\alpha,\beta,\,a,b}(I)$ onto itself. To show the continuity of $h^{\theta}_{1,\alpha,\beta,\,a,b}$, consider the sequence $\{x_n\}_{n\in\mathbb{N}}\to 0$ in $\mathcal{H}^{\theta}_{1,\alpha,\beta,\,a,b}(I)$ as $j\to\infty$. Using the above sequence in (2.4) and applying its convergence, one can easily verify the continuity of $h^{\theta}_{1,\alpha,\beta,\,a,b}$. Next, we prove the second part.

Following the above procedure, consider

$$(-1)^{k+q}(y\csc\theta)^{q/2}R^{*}_{1,\alpha,\beta,a+k-1,b,\theta}\dots R^{*}_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{-1}(y)$$

$$=e^{-i(k+q)\left(\frac{\pi}{2}-\theta\right)}\left(h^{\theta}_{1,\alpha,\beta,a+k+q,b}\right)^{-1}\left((\text{x}\csc\theta)^{\frac{k}{2}}R_{1,\alpha,\beta,a+q-1,b,\theta}\dots R_{1,\alpha,\beta,a,b,\theta}\right)(y).$$

Hence,

$$\left|y^{q}D_{y}^{k}y^{\frac{-\alpha+\beta-a+b}{2}}e^{\frac{i}{2}y^{2}\text{ü}\ \theta}\left(h^{\theta}_{1,\alpha,\beta,ak+q,b}\right)^{-1}(y)\right|$$

$$\leq\frac{\pi}{2}E'_{1,\alpha,\beta,a,b,\theta,k,q}\sum_{r=0}^{n+1}\binom{n+1}{r}\rho^{1,\alpha,\beta,a,b}_{r,q,\theta}(\varphi),$$

which shows the continuity of $\left(h^{\theta}_{1,\alpha,\beta,ak+q,b}\right)^{-1}$ and hence proved.

Similarly,

Theorem 2.4. The fractional power of the first generalized Hankel-Clifford type transformation $h^{\theta}_{2,\alpha,\beta,\,a,b}$ is an isomorphism from $\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}(I)$ to itself. Furthermore, $\left(h^{\theta}_{2,\alpha,\beta,\,a,b}\right)^{-1}$ is a linear and continuous map from $\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}(I)$ to itself.

We shall now provide operational formulas for the fractional power of the first generalized Hankel-Clifford type transformation $h^{\theta}_{1,\alpha,\beta,\,a,b}$.

Proposition 2.5. For $\alpha,\beta\in\mathbb{R},a\geq b$ and all $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$,

i. $R_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y)=e^{-i\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+1,b}\left(-(\text{x}\csc\theta)^{\frac{1}{2}}\varphi\right)(y),$

ii. $h^{\theta}_{1,\alpha,\beta,a+1,b}\left(R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)=-e^{i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y).$

Proof. For $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$, Let

$$R_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y)$$

$$=e^{\frac{i}{2}y^{2}\cot\theta}y^{\frac{\alpha+a-\beta-b+1}{2}}D_{y}y^{\frac{-\alpha-a+\beta+b}{2}}e^{-\frac{i}{2}y^{2}\cot\theta}y^{\alpha-\beta}\gamma_{a,b,\theta}\int_{0}^{\infty}C_{\alpha,\beta,a,b}$$

$$(\text{xy}\csc\theta)e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\cot\theta}\varphi(x)dx$$

$$=\gamma_{a\text{ü}b\ \theta}e^{-\frac{i}{2}y^{2}\text{ü}\ \theta}y^{\frac{\alpha+a-\beta-b+1}{2}}\int_{0}^{\infty}D_{y}\left[C_{a\ b}(\text{xy}\csc\theta)\right]$$

$$(\text{x}\csc\theta)^{-\frac{\alpha-a-\beta+b}{2}}e^{-\frac{i}{2}x^{2}\text{ü}\ \theta}\varphi(x)dx$$

$$=\gamma_{a,b,\theta}y^{\frac{\alpha+a-\beta-b+1}{2}}\int_{0}^{\infty}(\text{xy}\csc\theta)^{\frac{\alpha-a-\beta+b-1}{2}}C_{\alpha,\beta,a+1,b}$$

$$(\text{xy}\csc\theta)(-1)(\text{x}\csc\theta)^{1-\frac{\alpha-a-\beta+b}{2}}\times e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\cot\theta}\varphi(x)dx$$

$$=\gamma_{a,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}C_{\alpha,\beta,a+1,b}(\text{xy}\csc\theta)e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\text{ü}\ \theta}(-1)(\text{x}\csc\theta)^{\frac{1}{2}}\varphi(x)dx$$

$$=e^{-i\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a+1,b}\left(-(\text{x}\csc\theta)^{\frac{1}{2}}\varphi\right)(y).$$

For the second part, consider

$$h^{\theta}_{1,\alpha,\beta,a+1,b}\left(R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)=\gamma_{a+1,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}C_{\alpha,\beta,a+1,b}$$

$$(\text{xy}\csc\theta)e^{\frac{i}{2}y^{2}\cot\theta}x^{\frac{\alpha+a-\beta-b+1}{2}}D_{x}x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}x^{2}\cot\theta}\varphi(x)dx$$

$$=\gamma_{a+1,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}(\text{xy}\csc\theta)^{-\frac{\alpha-a-\beta+b-1}{2}}C_{\alpha,\beta,a+1,b}(\text{xy}\csc\theta)x^{\frac{\alpha+a-\beta-b+1}{2}}D_{x}x^{\frac{-\alpha-a+\beta+b}{2}}$$

$$\times e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\text{ü}\ \theta}\varphi(x)dx$$

$$=\gamma_{a+1,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}(y\csc\theta)^{-\frac{\alpha+a-\beta-b+1}{2}}(\text{xy}\csc\theta)^{a+1-b}C_{a+1,b}(\text{xy}\csc\theta)D_{x}x^{\frac{-\alpha-a+\beta+b}{2}}$$

$$\times e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\text{ü}\ \theta}\varphi(x)dx$$

$$=\gamma_{a+1,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}(y\csc\theta)^{-\frac{\alpha+a-\beta-b+1}{2}}(-1)(y\csc\theta)\left(\text{xy}\csc\theta)^{a-b}C_{a,b}(\text{xy}\csc\theta)\right)$$

$$\times x^{\frac{-\alpha-a+\beta+b}{2}}e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\text{ü}\ \theta}\varphi(x)dx$$

$$=-(y\csc\theta)^{\frac{1}{2}}e^{-i\left(\frac{\pi}{2}-\theta\right)}\gamma_{a,b,\theta}y^{\alpha-\beta}\int_{0}^{\infty}C_{\alpha,\beta,a+1,b}(\text{xy}\csc\theta)e^{\frac{i}{2}\left(x^{2}+y^{2}\right)\text{ü}\ \theta}\varphi(x)dx$$

$$=-e^{-i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\varphi\right)(y).$$

Hence, the theorem is proved.

Proposition 2.6. For $\alpha,\beta\in\mathbb{R},a\geq b$ and all $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a+1,b}(I)$,

i. $S_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\varphi\right)(y)=e^{-i\left(\frac{\pi}{2}-\theta\right)}h^{\theta}_{1,\alpha,\beta,a,b}\left((\text{x}\csc\theta)^{\frac{1}{2}}\varphi\right)(y),$

ii. $h^{\theta}_{1,\alpha,\beta,a,b}\left(S^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)=e^{i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\varphi\right)(y).$

Furthermore, For $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$,

iii. $h^{\theta}_{1,\alpha,\beta,a,b}\left(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)=-(y\csc\theta)h^{\theta}_{1,\alpha,\beta,a,b}\varphi(y),$

iv. $\Delta_{1,\alpha,\beta,a,b,\theta} h^{\theta}_{1,\alpha,\beta,a,b}\varphi(y) = h^{\theta}_{1,\alpha,\beta,a,b}\left(-(x\csc\theta)\varphi\right)(y).$

Proof. We leave these proofs to the readers as one can observe that (i) and (ii) can be derived using the techniques used in Proposition 2.5 Further, (iii) can be easily proved by combining Proposition 2.5 (ii) and Proposition 2.6 (ii), similarly, (iv) is easily derived by combining Proposition 2.5(i) and Proposition 2.6 (i). Hence proof is complete.

Here, we define some operational formulas for the fractional power of the second generalized Hankel-Clifford type transformation, analogous to the first.

Proposition 2.7. For $\alpha,\beta\in\mathbb{R}, a\ge b$ and all $\varphi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I)$,

i. $S_{2,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{2,\alpha,\beta,a,b}\varphi\right)(y)$

$$= e^{-i(\pi/2-\theta)} h^{\theta}_{2,\alpha,\beta,a+1,b}\left(-(x\csc\theta)^{1/2}\varphi\right)(y),$$

ii. $h^{\theta}_{2,\alpha,\beta,a+1,b}\left(S^{*}_{2,\alpha,\beta,a,b,\theta}\varphi\right)(y)$

$$= -e^{i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(h^{\theta}_{2,\alpha,\beta,a,b}\varphi\right)(y),$$

iii. $h^{\theta}_{2,\alpha,\beta,a,b}\left(\overline{\Delta}_{2,\alpha,\beta,a,b,\theta}\varphi\right)(y) = -(y\csc\theta)\, h^{\theta}_{2,\alpha,\beta,a,b}\varphi(y),$

iv. $\Delta_{2,\alpha,\beta,a,b,\theta} h^{\theta}_{2,\alpha,\beta,a,b}\varphi(y) = h^{\theta}_{2,\alpha,\beta,a,b}\left(-(x\csc\theta)\varphi\right)(y).$

For $\varphi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a+1,b}(I)$,

v. $R_{2,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{2,\alpha,\beta,a+1,b}\varphi\right)(y)$

$$= e^{-i\left(\frac{\pi}{2}-\theta\right)} h^{\theta}_{2,\alpha,\beta,a,b}\left((\mathrm{x}\csc\theta)^{\frac{1}{2}}\varphi\right)(y),$$

vi. $h^{\theta}_{2,\alpha,\beta,a,b}\left(R^{*}_{2,\alpha,\beta,a,b,\theta}\varphi\right)(y)$

$$= e^{i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(h^{\theta}_{2,\alpha,\beta,a+1,b}\varphi\right)(y).$$

Note that, similar results can be obtained for inverse transformations of $h^{\theta}_{1,\alpha,\beta,a,b}$ and $h^{\theta}_{2,\alpha,\beta,a,b}$. Using all these properties we define the fractional power of the first and second generalized Hankel Clifford transform on basic functions.

Fractional power of the first generalized Hankel Clifford transform on basic functions

1. $h^{\theta}_{1,\alpha,\beta,a,b}\left(\delta^{r}(x-c)\right) = (-1)^{r} D^{r}_{c} K^{\theta}_{1}(c,y)$.

2. $h^{\theta}_{1,\alpha,\beta,a,b}\left(x^{\frac{\alpha+a-\beta-b}{2}} H(c-x) e^{-\frac{i}{2}x^{2}\text{ü}\quad\theta}\right)$

$$= \gamma_{a,b,\theta} y^{\alpha-\beta} e^{-\frac{i}{2}y^{2}\cot\theta}(y\csc\theta)^{-1/2} c^{\frac{\alpha+a+1-\beta-b}{2}} C_{\alpha,\beta,a+1,b}(cy\csc\theta)$$

3. $h^{\theta}_{1,\alpha,\beta,a,b}\left(x^{\frac{\alpha+a-\beta-b}{2}+1} H(c-x) e^{-\frac{i}{2}x^{2}\cot\theta}\right)$

$$= \gamma_{a,b,\theta} y^{\alpha-\beta} e^{-\frac{i}{2}y^{2}\cot\theta}(y\csc\theta)^{-1/2} c^{\frac{\alpha+a+2-\beta-b}{2}}$$

$$\left[c^{1/2} C_{\alpha,\beta,a+1,b}(cy\csc\theta) - (y\csc\theta)^{-1/2} C_{\alpha,\beta,a+2,b}(cy\csc\theta)\right].$$

4. $h^{\theta}_{1,\alpha,\beta,a,b}\varphi(cx)$

$$= c^{\alpha-\beta-1}\left(\frac{\csc\theta_2}{\csc\theta_1}\right)^{\alpha-\beta} e^{\frac{i}{2}(a+1-b)(\theta_2-\theta_1)} e^{-\frac{i}{2}y^{2}\left[c^{2}-\left(\frac{\csc\theta_1}{\csc\theta_2}\right)^{2}\right]\cot\theta_2}\left(h^{\theta_2}_{1,\alpha,\beta,a,b}\varphi\right)\left(\frac{y\csc\theta_1}{\csc\theta_2}\right)$$

where $c^{2}\cot\theta_2 = \cot\theta_1$.

5. $h^{\theta}_{1,\alpha,\beta,a,b}\left(\frac{\varphi(x)}{\sqrt{x}}\right) = \frac{(y\csc\theta)^{\text{ü}}}{a-b}$

$$\left(e^{i(\pi/2-\theta)}\left(h^{\theta}_{1,\alpha,\beta,a-1,b}\varphi\right)(y) + e^{-i(\pi/2-\theta)}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\varphi\right)(y)\right)$$

6. $h^{\theta}_{1,\alpha,\beta,a,b}\left(\frac{\varphi(x)}{x}\right) = (y\csc\theta)$

$$\left(\frac{e^{i(\pi/2-\theta)}}{(a-b)^{2}+b-a}\left(h^{\theta}_{1,\alpha,\beta,a-2,b}\varphi\right)(y) + \frac{2}{(a-b)^{2}-1}\left(h^{\theta}_{1,\alpha,\beta,a,b}\varphi\right)(y)\right.$$

$$\left.+\frac{e^{-\text{ü}i(\pi\quad-\theta)}}{(a-b)^{2}+a-b}\left(h^{\theta}_{1,\alpha,\beta,a+2,b}\varphi\right)(y)\right)$$

7. $h^{\theta}_{1,\alpha,\beta,a,b}\left(D_{x}\varphi(x)\right) = -i\cot\theta\left(h^{\theta}_{1,\alpha,\beta,a,b}(x\varphi)\right)(y) + \frac{\alpha-\beta}{2}$

$$\left(h^{\theta}_{1,\alpha,\beta,a,b}\left(\frac{\varphi(x)}{x}\right)\right)(y) - \frac{(y\csc\theta)^{1/2}}{2}$$

$$\left[\begin{array}{l} e^{i(\pi/2-\theta)}\left(h^{\theta}_{1,\alpha,\beta,a-1,b}\left(\frac{\varphi(x)}{\sqrt{x}}\right)\right)(y) - e^{-i(\pi/2-\theta)} \\ \left(h^{\theta}_{1,\alpha,\beta,a+1,b}\left(\frac{\varphi(x)}{\sqrt{x}}\right)\right)(y) \end{array}\right]$$

Fractional power of the second generalized Hankel Clifford transform on basic functions

1. $h^{\theta}_{2,\alpha,\beta,a,b}\left(\delta^{r}\left(x-c\right)\right)=(-1)^{r}D^{r}_{c}K^{\theta}_{2}\left(c,y\right).$

2. $h^{\theta}_{2,\alpha,\beta,a,b}\left(x^{\frac{-\alpha+a+\beta-b}{2}}H\left(c-x\right)e^{-\frac{i}{2}x^{2}ü\ \ \theta}\right)$

$$=\gamma_{a,b,\theta}e^{\frac{i}{2}y^{2}\cot\theta}(y\csc\theta)^{-1/2}c^{\frac{\alpha+a+1-\beta-b}{2}}C_{\alpha,\beta,a+1,b}\left(cy\csc\theta\right).$$

3. $h^{\theta}_{2,\alpha,\beta,a,b}\left(x^{\frac{-\alpha+a+\beta-b}{2}+1}H\left(c-x\right)e^{-\frac{i}{2}x^{2}\cot\theta}\right)$

$$=\gamma_{a,b,\theta}e^{\frac{i}{2}y^{2}\cot\theta}(y\csc\theta)^{-1/2}c^{\frac{\alpha+a+2-\beta-b}{2}}$$

$$\left[c^{1/2}C_{\alpha,\beta,a+1,b}\left(cy\csc\theta\right)-(y\csc\theta)^{-1/2}C_{\alpha,\beta,a+2,b}\left(cy\csc\theta\right)\right].$$

4. $h^{\theta}_{2,\alpha,\beta,a,b}\varphi\left(cx\right)$

$$=c^{-(\alpha-\beta+1)}e^{\frac{i}{2}(a+1-b)\left(\theta_{2}-\theta_{1}\right)}e^{\frac{i}{2}y^{2}\left[c^{2}-\left(\frac{\csc\theta_{1}}{c\csc\theta_{2}}\right)^{2}\right]\cot\theta_{2}}$$

$$\left(h^{\theta_{2}}_{2,\alpha,\beta,a,b}\varphi\right)\left(\frac{y\csc\theta_{1}}{c\csc\theta_{2}}\right)$$

where $c^{2}\cot\theta_{2}=\cot\theta_{1}.$

5. $h^{\theta}_{2,\alpha,\beta,a,b}\left(\frac{\varphi\left(x\right)}{\sqrt{x}}\right)=\frac{(y\csc\theta)^{ü}}{a-b}$

$$\left(e^{i(\pi/2-\theta)}\left(h^{\theta}_{2,\alpha,\beta,a-1,b}\varphi\right)\left(y\right)+e^{-i(\pi/2-\theta)}\left(h^{\theta}_{2,\alpha,\beta,a+1,b}\varphi\right)\left(y\right)\right)$$

6.. $h^{\theta}_{2,\alpha,\beta,a,b}\left(\frac{\varphi\left(x\right)}{x}\right)=\left(y\csc\theta\right)$

$$\left(\frac{e^{i(\pi/2-\theta)}}{(a-b)^{2}+b-a}\left(h^{\theta}_{2,\alpha,\beta,a-2,b}\varphi\right)\left(y\right)+\frac{2}{(a-b)^{2}-1}\left(h^{\theta}_{2,\alpha,\beta,a,b}\varphi\right)\left(y\right)\right.$$

$$\left.+\frac{e^{-üi(\pi\ \ -\theta)}}{(a-b)^{2}+a-b}\left(h^{\theta}_{2,\alpha,\beta,a+2,b}\varphi\right)\left(y\right)\right).$$

7. $h^{\theta}_{2,\alpha,\beta,a,b}\left(D_{x}\varphi\left(x\right)\right)=-i\cot\theta$

$$\left(h^{\theta}_{2,\alpha,\beta,a,b}\left(x\varphi\right)\right)\left(y\right)+\frac{\alpha-\beta}{2}\left(h^{\theta}_{2,\alpha,\beta,a,b}\left(\frac{\varphi\left(x\right)}{x}\right)\right)\left(y\right)-\frac{(y\csc\theta)^{1/2}}{2}$$

$$\left[\begin{array}{l}e^{i(\pi/2-\theta)}\left(h^{\theta}_{2,\alpha,\beta,a-1,b}\left(\frac{\varphi\left(x\right)}{\sqrt{x}}\right)\right)\\ \left(y\right)-e^{-i(\pi/2-\theta)}\left(h^{\theta}_{2,\alpha,\beta,a+1,b}\left(\frac{\varphi\left(x\right)}{\sqrt{x}}\right)\right)\left(y\right)\end{array}\right]$$

The generalized Fractional power of generalized Hankel Clifford transformations

This section introduces the generalized Fractional power of generalized Hankel Clifford transformations and their characteristics. Note that in this section, we will work with $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)$, as one can derive all the analogous results for $\left(h^{\theta}_{2,\alpha,\beta,a,b}\right)'$ using this section.

Definition 3.1. For $\alpha,\beta\in\mathbb{R},a\geq b$, the generalized Fractional power of the first generalized Hankel Clifford transformation $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)$ on $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)\left(I\right)$ as the adjoint operator of $h^{\theta}_{2,\alpha,\beta,a,b}$ on $\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\left(I\right)$ as

$$\left\langle\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'f,\Phi\right\rangle$$

$$=\left\langle f,h^{\theta}_{2,\alpha,\beta,a,b}\Phi\right\rangle,\forall f\in\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)'\left(I\right);\Phi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\left(I\right).\#(3.1)$$

Observe that (3.1) can be treated as generalized mixed Parseval's equation as if we suppose $\Phi=\overline{h^{\theta}_{2,\alpha,\beta,a,b}\varphi},\varphi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\left(I\right)$ then (ü) becomes

$$\left\langle\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'f,\overline{h^{\theta}_{2,\alpha,\beta,a,b}\varphi}\right\rangle=\left\langle f,h^{\theta}_{2,\alpha,\beta,a,b}\left(\overline{h^{\theta}_{2,\alpha,\beta,a,b}\varphi}\right)\right\rangle$$

$$=\left\langle f,h^{\theta}_{2,\alpha,\beta,a,b}\left(h^{\theta}_{2,\alpha,\beta,a,b}\right)^{-1}\overline{\varphi}\right\rangle=f,\overline{\varphi}.$$

Special Case:

Note that the Fractional power of the first generalized Hankel Clifford transformation $h^{\theta}_{1,\alpha,\beta,a,b}$ is a particular case of the generalized Fractional power of the first generalized Hankel Clifford transformation $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)$ as by (1.23), we know that $\mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}\left(I\right)\subset\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)'\left(I\right)$ and using mixed Parseval's equation (1.11), we have

$$\left\langle\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'f,\Phi\right\rangle=\left\langle h^{\theta}_{1,\alpha,\beta,a,b}f,\Phi\right\rangle,\Phi\in\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\left(I\right).$$

Which can be written as

$$\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)' f = h^{\theta}_{1,\alpha,\beta,\,a,b} f.$$

Definition 3.2. For $\alpha,\beta \in \mathbb{R},\ a \geq b$, the inverse generalized Fractional power of the first generalized Hankel Clifford transformation $\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)'^{-1}$ on $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}\right)'(I)$ as

$$\left\langle \left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{\text{ü-}} f,\Phi \right\rangle = \left\langle f,\left(h^{\theta}_{2,\alpha,\beta,a,b}\right)^{-} \Phi \right\rangle, \forall \Phi \in \mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I).\#(3.2)$$

Theorem 3.3. The generalized fractional power of the first generalized Hankel-Clifford type transformation $\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)'$ is an isomorphism from $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}\right)'(I)$ to itself.

Proof. For $\Phi \in \mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I)$ then by Theorem 2.4, . $h^{\theta}_{1,\alpha,\beta,a,b}\Phi \in \mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I)$ Which implies that (3.1) is well-defined. Clearly, it is linear. To show the continuity, consider a sequence $\left\{x_n\right\}_{n\in\mathbb{N}} \to 0$ in $\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}(I)$ as $j \to \infty$. As we know that $h^{\theta}_{2,\alpha,\beta,\,a,b}$ is continuous, then $\left\{h^{\theta}_{2,\alpha,\beta,\,a,b}x_n\right\}_{n\in\mathbb{N}} \to 0$ in $\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}(I)$ as $j \to \infty$. Hence from (3.1) $\left\{\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)' f, x_n\right\}_{n\in\mathbb{N}} \to 0$ in $\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}(I)$ as $j \to \infty$.

Therefore, $\left(h^{\theta}_{1,\alpha,\beta,\,a,b}\right)'$ is continuous on $\left(\mathcal{H}^{\theta}_{2,\alpha,\beta,\,a,b}\right)'(I)$.

From (3.2), one can see $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{\text{ü-}} f \in \left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)(I)$

, Hence by using (3.1) and (3.2), we have,

$$\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'^{-1} f = f = \left(h^{\theta}_{1,\alpha,\beta,a,b}\right)^{-1}\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)' f.$$

Therefore, we conclude that the proof is complete.

Now, we state similar operational formulas as we saw in section 4 for the generalized fractional power of the first generalized Hankel-Clifford type transformation.

Proposition 3.4. For $\alpha,\beta \in \mathbb{R}, a \geq b$ and all $\varphi \in \left(\mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}\right)'(I)$,

i. $R_{1,\alpha,\beta,a,b,\theta}\left(\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\varphi\right)(y)$

$$= e^{-i(\pi/2-\theta)}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\right)'\left(-(x\csc\theta)^{1/2}\varphi\right)(y)$$

ii. $\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\right)'\left(R^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$

$$= -e^{i(\pi/2-\theta)}(y\csc\theta)^{1/2}\left(\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\varphi\right)(y)$$

iii. $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\left(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$

$$= -\left(y\csc\theta\right)\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\varphi(y)$$

iv. $\Delta_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\varphi(y)$

$$= \left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\left(-\left(x\csc\theta\right)\varphi\right)(y)$$

For $\varphi \in \left(\mathcal{H}^{\theta}_{1,\alpha,\beta,a+1,b}\right)'(I)$,

v. $S_{1,\alpha,\beta,a,b,\theta}\left(h^{\theta}_{1,\alpha,\beta,a+1,b}\right)'\varphi(y)$

$$= e^{-i\left(\frac{\pi}{2}-\theta\right)}\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\left((x\csc\theta)^{\frac{1}{2}}\varphi\right)(y),$$

vi.. $\left(h^{\theta}_{1,\alpha,\beta,a,b}\right)'\left(S^{*}_{1,\alpha,\beta,a,b,\theta}\varphi\right)(y)$

$$= e^{i\left(\frac{\pi}{2}-\theta\right)}(y\csc\theta)^{\frac{1}{2}}\left(\left(h^{\theta}_{2,\alpha,\beta,a+1,b}\right)'\varphi\right)(y).$$

Remark. The study is motivated by (Prasad & Kumar, 2014), and we firmly believe that the result obtained in our research is more vital than it and can be reduced to (Prasad & Kumar, 2014) by putting the value $\alpha = \frac{1}{4}+\frac{\mu}{2}, \beta = \frac{1}{4}-\frac{\mu}{2}$ and $a = \frac{1}{4}+\frac{v}{2}, b = \frac{1}{4}-\frac{v}{2}$.

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Conflict of Interest

The authors declare no competing interests related to A Pair of Fractional Power of Generalized Hankel-Clifford Type Transformations and Their Characteristics.

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