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ORIGINAL RESEARCH PAPER

Eco-epidemiology of prey and competitive predator species in the SEI model

R. Sivakumar*, **S. Vijaya**

Abstract

The ecological epidemiology structure explores the relationship between disease and competitiveness in a predator-prey [22] (Vijaya S, J. J, 2017) structure. We create a mathematical model that includes a susceptible (S), exposed (E), and infected (I)[28][22] (Vijaya S, J.J, 2017)(S.P.Bera, A. M, 2015) subpopulation of prey, as well as a competing predator. The model Examines how disease transmission, predation rates, and natural population dynamics affect structure stability. The findings provide insights into illness prevalence and population levels, which could help researchers better understand disease outbreaks and the function of predators in disease control. Further studies should examine spatial aspects, environmental consequences, and predator behaviors.

Keywords: Eco-epidemiological, Susceptible exposed infected model, Predator-prey relationship, Disease transmission, Population dynamics.

MSC: 92D30 (Mathematical Models in Population Biology), 92E05 (stability of dynamical structures), 34C25 (Ordinary differential equations with discontinued right-hand sides)

Introduction

Eco-epidemiology examines the relationship between ecological relationships and disease dynamics within a community. In this sense, a predator-prey mechanism affected by an illness in the prey population presents an intriguing topic of study. This study looks at a scenario in which a contagious disease divides a prey species into three subpopulations: susceptible (S), exposed (E), and infected (I) (Vijaya S, J. J, 2017)(S. P. Bera, A. M, 2015). The predator species is considered a competitor, preying on susceptible, exposed, and infected people.

The purpose of this project is to create a mathematical model that represents the population Mechanism of susceptible, exposed, and infected prey and predator species (Vijaya S. J. J, 2017; Samin Akhtar, Sahabuddin, 2021). This model will include disease transmission rate, predation rate, and natural growth/mortality rates for each population. By analyzing the model, we want to learn how the existence of the disease and the predator population affect each other and the overall stability of the structure (Vijaya S. J. J, 2016; Gimmelli, Giacomo, 2015).

Prey species are classified into three sub-populations: susceptible (S), exposed (E), and infected (I) (Vijaya S. J. J, 2017; Vijaya S. J. J, 2016). Individuals with characteristics susceptible to the disease can contract it, become exposed, and become infected. The illness may impact the prey's growth rate and survival. Predator species compete for the same prey habitats. The predator's predation rate may change between susceptible and infected prey, depending on predator preference and prey sensitivities.

This introduction examined the disease's peculiar features, predator functional responses (how predation rate varies with prey density), and the potential consequences of this complex relationship between ecology and epidemiology. This form is a susceptible-exposed-infected relationship employing Holling type II functional response (Nawaj Sarif & Sahabuddin Sawardi, 2021) using various working methods, which is quite intriguing.

Mathematical Modelling

The organized model is formed in the same way as the mathematical model. In this section, some basic

Department of Mathematics, Annamalai University, Annamalai Nagar, Chidambaram, Tamil Nadu, India.

***Corresponding Author:** R. Sivakumar, Department of Mathematics, Annamalai University, Annamalai Nagar, Chidambaram, Tamil Nadu, India, E-Mail: sivamathematics1729@gmail.com

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assumptions are made: (i) Let x represent the population density of the prey, y represent the population density of the first susceptible predator, E represent the population density of the third exposed predator, and z represent the population density of the second infected predator at time t.

$$\frac{dx}{dt} = x\left(r - ax - c_1 y - c_2 z - c_4 E\right)$$

$$\frac{dE}{dt} = E\left(\beta y + \alpha z - c_4 E + d_2\right)$$

$$\frac{dy}{dt} = y\left(-\alpha z - c_4 z + c_3 x - d_1\right)$$

$$\frac{dz}{dt} = z\left(\alpha y + c_5 x - c_6 y - d_2\right) \qquad \text{…………. (2.1)}$$

With beginning circumstances.

$$\mathrm{X}(0) \geq 0, E(0) \geq 0, y(0) \geq 0, z(0) \geq 0$$

Some basic assumptions are

In time t, x represents the population density of the prey, y represents the population density of the first Susceptible predator, E represents the population density of the third exposed predator, and z represents the population density of the second infected predator.

- r→ represents the prey species' inherent growth rate.
- a→ represents the rate of competing prey species.
- c_1, c_2 are The rate at which the susceptible and infected predators capture prey, respectively (Vijaya S & Rekha E, 2017) (Vijaya S & J. J, 2017) (S. P. Bera, A. M, 2015)
- c_3, c_5 The conversion rates for the susceptible and infected predators as a result of consuming prey (Vijaya S & Rekha E, 2017; Vijaya S & J. J, 2017; S. P. Bera, A. M, 2015)
- d_1, d_2 represent overcrowding in the susceptible and diseased predators, respectively (Sabah Ali Rahi & Raid Kamel Naji, 2024).
- c_4 is the speed at which the susceptible predator is captured by the infected predator (Xin-You Meng, Ni-Ni Qin, Hai-Feng Huo, 2021).
- c_6 Indicates the rate at which the infected predator is captured by the susceptible predator (Xin-You Meng, Ni-Ni Qin, Hai-Feng Huo, 2021).
- α represents the power of infection between the diseased and susceptible predators.
- β is the rate at which susceptible individuals (s) get exposed (E) through contact with infected individuals.

Positiveness and Bounded of Theorems

Theorem 3.1

The equations (2.1) are always non-negative. Then, all possible structure (Vijaya S & J. J, 2017) solutions (2.1) were positive.

Considering the very first equation (2.1) of the entire structure, we obtain

$\frac{dx}{dt}$ = x(r-ax-c_1 y-c_2 z-c_4 E)

$\frac{dx}{dt}$ = Φ (x,y,z)dt………………(3.1) (Vijaya S & Rekha E,2016)

Where

Φ(x,y,z) = x(r-ax-c_1 y-c_2 z-c_4 E) (Vijaya S & Rekha E, 2016; Russell L.Herman, 2013)

Taking integrating in the area [0,t] yields (Vijaya S & J. J, 2017)

X(t)= x(0) $e^{\int \text{Ö}(x,y,z)\,dt} > 0 \,\forall\, t \; as \; x(0) \geq 0$………..(3.2) (Victor Henner, Tatyana Belozerova & Mikhail Khenner, 2019)

Next, take the 2nd set of equations (2.1) structure, where (Vijaya S & J.J,2017)

$$\frac{dE}{dt} = E\left(\beta y + \alpha z - c_4 E + d_2\right)$$

$$\frac{dE}{dt} = \gamma\left(y, z\right)dt \text{………………..}(3.3)$$

Where

$$\gamma\left(y, z\right)dt = E\left(\beta y + \alpha z - c_4 E + d_2\right)$$

Taking integrating in the area [0,t] yields (Vijaya S & J.J,2017)

E (t) = E(0) $e^{\int \text{ã}(y,z)\,dt} > 0 \,\forall\, t \; as \; E(0) \geq 0$…………(3.4)

Next, take the 3rd set of equations (2.1) structure whereby (Vijaya S & J.J,2017)

$$\frac{dy}{dt} = y\left(-\alpha z - c_4 z + c_3 x - d_1\right)$$

$$\frac{dy}{dt} = \varphi\left(x, z\right)dt \text{………………..}(3.5)$$

Where

$$\varphi\left(x, z\right)dt = y\left(-\alpha z - c_4 z + c_3 x - d_1\right)$$

Taking integrating in the area [0,t] yields[22] (Vijaya S & J.J,2017)

Y(t) = Y(0) $e^{\int \text{ö}(x,z)\,dt} > 0 \,\forall\, t \; as \; Y(0) \geq 0$…………(3.6)

Next, take the 4th set of equation (2,1) structure whereby [22] (Vijaya S & J.J,2017)

$$\frac{dz}{dt} = z\left(\alpha y + c_5 x - c_6 y - d_2\right)$$

$$\frac{dz}{dt} = \div\left(\mathrm{x},\mathrm{y}\right)\mathrm{dt}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...(3.7)$$

Where

$$\div\left(\mathrm{x},\mathrm{y}\right) = \quad z\left(\alpha y + c_5 x - c_6 y - d_2\right)$$

Taking integrating in the area [0,t] yields[22] (Vijaya S & J.J,2017)

Z(t)= Z(0) $e^{\int \div (\mathrm{x},\mathrm{y})\,dt} > 0\,\forall\, t\ as\ Z\left(0\right) \geq 0\ldots\ldots\ldots\ldots(3.8)$ (Vijaya S & Rekha E,2016)

As the outcome, we can conclude that all structure (2.1) results are always positive.

Theorem 3.2

The structure trajectory (2.1) is bound.

Let (I = X + E + Y + Z) and examine its time derivative along the solution path of (2.1)[22] (Vijaya S & J.J,2017) (Vijaya S & Rekha E,2017)

$$\frac{dl}{dt} = \frac{dx}{dt} + \frac{dE}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$$

Now $\frac{dl}{dt} + \rho l \leq rx - a_1 x^2 + \rho x + \rho E + \rho y + \rho z - zd_1 - zd_2$

Where

$\rho\, is\ a\ positive constant\ for\ r_1 + \rho - a_1 x \geq 0\ \rho - d_1 \geq 0\ \rho - d_2 \geq 0\ given \in > 0.$

There exists to such that t$\geq t_0$

$$\frac{dl}{dt} + \rho l \leq m + \in, if\ m = \min\left\{\frac{\rho + r_1}{a_1}, \rho - d_1, \rho - d_2\right\}$$

Hence $\frac{d}{dt}\left(le^{\rho l}\right) \leq \left(m + \in\right)e^{\rho t}$

$$\Rightarrow l\left(t\right) \leq l\left(t_0\right)e^{-\rho\left(t - t_0\right) + \frac{(m+t)}{\rho}}(1 - e^{-l\left(t - t_0\right)}$$

putting t$\rightarrow$ 0 then letting $\in \rightarrow 0$

$$\limsup_{t\rightarrow\infty} l\left(t\right) \leq \frac{m}{\rho}$$

On the starting condition, the structure (2.1) is bound.

Analytical solution of critical points

The steady-state equations[22] (S. Vijaya, J. J, 2017) provide the equilibrium point for the parametric model (S. P. Bera, A. M, 2015) (2.1).

$\frac{dx}{dt} = \frac{dE}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$. After performing the calculation, we obtain both the trivial and non-trivial equilibrium points (S. P. Bera, A. M, 2015) (Vijaya S & Rekha E, 2017).

- The trivial equilibrium (S. Vijaya, J. J, 2017) points of the {x=0, E=0, y=0, z=0} this all prey-predator absents state of equilibrium always exists.
- Both susceptible-infected predator equilibrium points { x = $\frac{r}{a}, E = 0, y = 0, z = 0\}$this equilibrium point prey is available, susceptible predator, Exposed predator, and infected predator are unavailable.
- Infected predator-free equilibrium point {x= $\frac{d_1}{c_3}, E = 0, y = \frac{rc_3 - ad_1}{c_1 c_3}, z = 0\}$ this equilibrium point, prey is present susceptible predator is available and Exposed, and the infected predator is unavailable.
- Susceptible predator-free equilibrium point {x= $\frac{d_2}{c_5}, E = 0, y = 0, z = \frac{rc_5 - ad_2}{c_2 c_5}\}$ this equilibrium point: prey is available, susceptible and Exposed predator is unavailable and infected predator is present (Vijaya S & Rekha E, 2016)
- Exposed predator-free equilibrium point {x= $\frac{d_2}{c_3}, E = \frac{\beta y + d_2}{c_4}, y = 0, z = 0\}$ this equilibrium point prey is present exposed predator is available and susceptible, and infected predator is unavailable.
- Interior equilibrium points are {x = $x^*, E = E^*$, y = $y^*, z = z^*\}$

Where,

$$x = \frac{rc_3 - ad_1}{c_3} - \frac{rc_1 c_3 - ac_1 d_1}{c_1 c_3} - \frac{rc_3 c_2 - ac_2 d_1}{c_2 c_3} - \frac{\beta c_4 y + d_2 c_4}{c_4}$$

$$E = \frac{\beta rc_3 - \beta ad_1}{c_1 c_3} + \frac{\alpha rc_3 - ad_1}{c_2 c_3} - \frac{rc_4 c_3 - ad_1}{c_1 c_3} + d_2$$

$$\mathrm{Y} = \frac{-\alpha rc_3 + \alpha ad_1}{c_2 c_3} - \frac{rc_4 c_3 - c_4 ad_1}{c_2 c_3}$$

$$\mathrm{Z} = \frac{\alpha rc_3 - \alpha ad_1}{c_1 c_3} - \frac{c_5 d_1}{c_3} - \frac{c_6 rc_3 - c_6 ad_1}{c_1 c_3} - d_2$$

The structure of the nonlinear differential equation (2.1) of Jacobian matrices is (S.Vijaya, J. J, 2017)

$$\mathrm{J} = \begin{bmatrix} -2ax - c_1 y - c_2 z - c_4 E + r & -c_1 x & -c_2 x & 0 \\ c_3 x & -\alpha z - c_4 z - d_1 & \alpha y + c_4 z - c_6 y + \beta y & \beta y + \alpha z \\ 0 & \alpha y & c_5 z - c_6 z & \alpha z - c_4 y \\ 0 & \beta E & \alpha E & \beta y + \alpha z - c_4 y - d_2 \end{bmatrix}$$

Structure of the Stability Analyser

Theorem 5.1

The trivial equilibrium point (0,0,0,0) of the structure (2.1) is a saddle point (Vijaya S & Rekha E, 2017; S. Vijaya, J. J, 2017).

Proof

The variance for the Jacobian matrices is (Vijaya S & Rekha E, 2017)

$$J_1 = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_2 \end{bmatrix}$$

"The eigenvalues are $\lambda_1 = r > 0, \lambda_2 = -d_1 < 0, \lambda_3 = 0, \lambda_4 = -d_2$ <0. One eigenvalue is positive and two are negative with condition $r > 0, -d_1 < 0, -d_2 < 0$. Therefore, the state of the equilibrium point (0,0,0,0) is called a saddle point"(Ranjit Kumar Upadhyay & Satteluri R. K Iyengar, 2019; S.Vijaya, J. J, 2017)

Theorem 5.2

Both Susceptible -Infected predator-free equilibrium point {x= $\frac{r}{a}, E=0, y=0, z=0\}$

of the structure (2.1) is stable, provided that a(Nayyereh Babakordi, Hamid R. Z. Zangeneh & Mojtaba Mostafavi Ghahfarokhi, 2019)

Proof

The variant for the Jacobian matrices of

$$J_2 = \begin{bmatrix} -r & \frac{-rc_1}{a} & \frac{-rc_2}{a} & 0 \\ \frac{rc_3}{a} & d_1 & \frac{rc_5}{a} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \end{bmatrix}$$

So eigenvalues are $\lambda_1 = -r\langle 0, \lambda_2 = d_1 \rangle 0, \lambda_3 = 0, \lambda_4 = d_2 > 0$. only if r>0, a $d_1 > cr_3$,$ad_2 > c_5 r$. Therefore, the given equilibrium point is stable.

Theorem 5.3

Infected predator-free equilibrium point {x= $\frac{d_1}{c_3}, E=0, y=\frac{rc_3 - ad_1}{c_1c_3}, z=0\}$ of the structure of equation (2.1) is locally asymptotically Secure, assuming that

a $\alpha d_1 + c_1c_3d_2 + rc_3c_6 > ac_6d_1 + \alpha rc_3 + c_1c_5d_{1.}$

Proof

The variations of the Jacobian matrices are

$$J_3 = \begin{bmatrix} \frac{-2ad_1}{c_3} - \frac{ad_{1-rc_3}}{c_3} + r & \frac{-c_1d_1}{c_3} & \frac{-d_1c_2}{c_3} & 0 \\ d_1 & d_1 & \alpha\left(\frac{ad_1 - rc_3}{c_1c_3}\right) - c_6\left(\frac{ad_1 - rc_3}{c_1c_3}\right) & \beta\left(\frac{ad_1 - rc_3}{c_1c_3}\right) \\ 0 & \alpha\left(\frac{ad_1 - rc_3}{c_1c_3}\right) & 0 & -c_4\left(\frac{ad_1 - rc_3}{c_1c_3}\right) \\ 0 & 0 & 0 & \beta\left(\frac{ad_1 - rc_3}{c_1c_3}\right) - c_4\left(\frac{ad_1 - rc_3}{c_1c_3}\right) \end{bmatrix}$$

The corresponding eigenvalues are:

The eigenvalues are $\lambda_1 = \frac{-2ad_1}{c_3} - \frac{ad_{1-rc_3}}{c_3} + r > 0, \lambda_2 = d_1 > 0, \lambda_3 = 0, \lambda_4 = \beta\left(\frac{ad_1 - rc_3}{c_1c_3}\right) - c_4\left(\frac{ad_1 - rc_3}{c_1c_3}\right). hence\, \lambda_1 \text{ and } \lambda_2 \text{ have negative real parts}. \lambda_3 = 0 \text{ with the condition}$

that a $\alpha d_1 + c_1c_3d_2 + rc_3c_6 > ac_6d_1 + \alpha rc_3 + c_1c_5d_{1.}$

The structure is locally asymptotically stable.

Theorem 5.4

Susceptible predator-free equilibrium points

{x = $\frac{d_2}{c_5}, E=0, y=0, z=\frac{rc_5 - ad_2}{c_2c_5}\}$

is locally asymptotically stable, assuming that a

$\alpha d_2 + ac_4d_2 + c_2c_3d_2 > rc_5 + c_4rc_5 + c_2c_5d_{1.}$

Proof

The variant for the Jacobian matrices is

$$J_4 = \begin{bmatrix} \frac{-2ad_2}{c_5} + \frac{ad_2 - c_{5r}}{c_5} + r & \frac{-d_2c_1}{c_5} & \frac{-d_2c_2}{c_5} & 0 \\ \frac{c_3d_2}{c_5} & -\alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) - c_4\left(\frac{rc_5 - ad_2}{c_2c_5}\right) & d_2 & \alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) \\ 0 & 0 & c_5\left(\frac{rc_5 - ad_2}{c_2c_5}\right) & \alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) \\ 0 & 0 & 0 & \alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) + d_2 \end{bmatrix}$$

The corresponding eigen values are $\lambda_1 = \frac{-2ad_2}{c_5} + \frac{ad_2 - c_{5r}}{c_5} + r < 0, \lambda_2 = -\alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) - c_4\left(\frac{rc_5 - ad_2}{c_2c_5}\right) < 0, \lambda_3 = 0, \lambda_4 = \alpha\left(\frac{rc_5 - ad_2}{c_2c_5}\right) d_2$ <0. Here λ_1, λ_2 *has negative real parts*.$\lambda_3 = 0$ *with the condition* that that a $\alpha d_2 + ac_4d_2 + c_2c_3d_2 > rc_5 + c_4rc_5 + c_2c_5d_{1.}$ ∴ The structure is locally asymptotically stable.

Theorem 5.5

Susceptible predator-free equilibrium point {x= $\frac{d_2}{c_3}$,$E = \frac{\beta y + d_2}{c_4}, y=0, z=0\}$ *is locally asymptotically stable* .

Proof:

The variants for the Jacobian matrices are

$$J_5 = \begin{bmatrix} \frac{-2ad_2}{c_3} - c_4\left(\frac{\beta y + d_2}{c_4}\right) + r & \frac{-c_1d_2}{c_3} & \frac{-c_2d_2}{c_3} & 0 \\ \frac{c_3d_2}{c_3} & d_1 & \frac{c_5d_2}{c_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta\left(\frac{\beta y + d_2}{c_4}\right) & \alpha\left(\frac{\beta y + d_2}{c_4}\right) & d_2 \end{bmatrix}$$

The corresponding eigenvalues are

$$\lambda_1 = \frac{-2ad_2}{c_3} - c_4\left(\frac{\beta y + d_2}{c_4}\right) + r\langle 0, \lambda_2 = d_1 \rangle 0, \lambda_3 = 0, \lambda_4 = d_2 > 0$$

$$\lambda_1 = \frac{-2ad_2}{c_3} + \beta y + d_2 + r\langle 0. \lambda_2 = d_1 \rangle 0, \lambda_3 = 0, \lambda_4 = d_2 > 0$$

Therefore, the structure is stable in the local asymptotic sense (Santosh Biswas, Sudip Samanta & Joydev Chattopadhyay, 2017)

Theorem 5.6

The equilibrium point within the interior{x=x*, E=E*, y=y*, z=z*} is asymptotically stable in the local sense (Vijaya S & Rekha E, 2016).

Proof

The variations of the Jacobian matrices are (Vijaya S & Rekha E, 2016)

$$J_6 = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}$$

Where,

$$N_{11} = -2a\left(\frac{rc_3 - ad_1}{c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) - \left(\frac{\beta c_4 y + d_2c_4}{c_4}\right) -$$

$$c_1\left(\frac{-\alpha rc_3 + \alpha ad_1 - rc_4c_3 - c_4ad_1}{c_2c_3}\right) - c_2\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} \quad -$$

$$d_2) - c_4\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + \beta\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + \alpha\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + d_{2.}$$

$$N_{12} = -c_1\left(\frac{rc_3 - ad_1}{c_1c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) - \left(\frac{\beta c_4 y + d_2c_4}{c_4}\right)$$

$$N_{13} = -c_2\left(\frac{rc_3 - ad_1}{c_1c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) - \left(\frac{\beta c_4 y + d_2c_4}{c_4}\right)$$

$$N_{14} = 0$$

$$N_{21} = c_3\left(\frac{rc_3 - ad_1}{c_1c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) - \left(\frac{\beta c_4 y + d_2c_4}{c_4}\right)$$

$$N_{22} = -\alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) - c_4\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)$$

$$N_{23} = \alpha\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right) + c_5\left(\frac{rc_3 - ad_1}{c_1c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \right.$$

$$\left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) - \left(\frac{\beta c_4 y + d_2c_4}{c_4}\right) - c_6\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right)$$

$$\left. + \beta\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right)\right)$$

$$N_{24} = \beta\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right) + \alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \right.\right.$$

$$\left.\left.\frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)\right)$$

$$N_{31} = 0$$

$$N_{32} = \alpha\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right)$$

$$N_{33} = c_5\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_6\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)$$

$$N_{34} = \alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_4\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)$$

$$N_{41} = 0$$

$$N_{42} = \beta\left(\frac{\beta rc_3 - \beta ad_1}{c_1c_3} + \frac{\alpha rc_3 - ad_1}{c_2c_3} - \frac{rc_4c_3 - ad_1}{c_1c_3} + d_2\right)$$

$$N_{43} = \alpha\left(\frac{\beta rc_3 - \beta ad_1}{c_1c_3} + \frac{\alpha rc_3 - ad_1}{c_2c_3} - \frac{rc_4c_3 - ad_1}{c_1c_3} + d_2\right)$$

$$N_{44} = \beta\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right) +$$

$$\alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_4\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right) + d_2$$

The corresponding eigenvalues are

$$\lambda_{1=} - 2a\left(\frac{rc_3 - ad_1}{c_3}\right) - \left(\frac{rc_1c_3 - ac_1d_1}{c_1c_3}\right) - \left(\frac{rc_2c_3 - ac_2d_1}{c_2c_3}\right) -$$

$$\left(\frac{\beta c_4 y + d_2c_4}{c_4}\right) - c_1\left(\frac{-\alpha rc_3 + \alpha ad_1 - rc_4c_3 - c_4ad_1}{c_2c_3}\right) -$$

$$c_2\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} -$$

$$d_2) - c_4\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + \beta\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + \alpha\left(\frac{rc_3 - ad_1}{c_1c_3}\right) + d_{2.}$$

$$\lambda_2 = -\alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_4\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)$$

$$\lambda_3 = c_5\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_6\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right)$$

$$\lambda_4 = \beta\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right) +$$

$$\alpha\left(\left(\frac{\alpha rc_3 - \alpha ad_1}{c_1c_3}\right) + \frac{c_5d_1}{c_3} - \frac{c_6rc_3 - c_6ad_1}{c_1c_3} - d_2\right) -$$

$$c_4\left(\left(\frac{-\alpha rc_3 + \alpha ad_1}{c_2c_3}\right) - \left(\frac{rc_4c_3 - ad_1c_4}{c_2c_3}\right)\right) + d_2$$

The characteristic equation is $\wedge_1(\lambda) = B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4$

Where,

$B_1 = 1$, $B_2 = -(m_{33} + m_{22} + m_{11})$, $B_3 = -(-m_{11}m_{22} - m_{11}m_{33} + m_{12}m_{21} + m_{13}m_{31} - m_{22}m_{33} + m_{23}m_{32})$,

$$B_4 = m_{11}m_{22}m_{33} + m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33}$$

$$- m_{13}m_{23}m_{31} - m_{13}m_{21}m_{32} + m_{13}m_{22}m_{31.}$$

Routh Hurwitz criteria, all the eigenvalues[22] of J_6 contain elements with negative real parts (Vijaya S & Rekha E,2017)

$B_1 > 0$

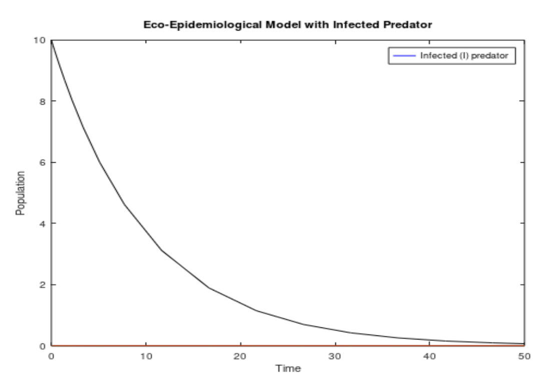
- $B_3>0$
- $B_1B_2B_3> B_3^2+ B_1^2B_4$.

Thus, given the structure of nonlinear differential equations (2.1) is locally stable near the non-trivial equilibrium (Saranya S, Vijaya S,2022)(point {x=x*, E=E*, y=y*, z=z*} if conditions are stated (Smith KF, A.-W. K, 2009).

Computational Modeling

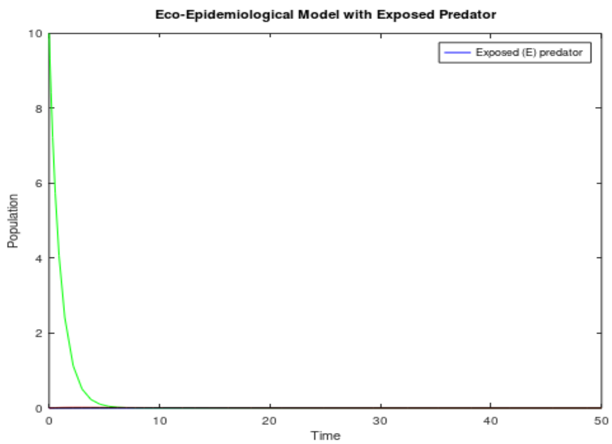
The form of the nonlinear differential equation (2.1) for numerical solution.

- First we determine the variables of the structure[33] as $\rho_1 = (\alpha = 1, r = 10, a = 1, c_1 = 1, c_2 = 1, c_3 = 12, c_4 = 1, c_5 = 10, c_6 = 1, d_1 = 1, d_2 = 1)$. Then the starting condition satisfied (x(0)=0, E(0)=0, y(0)=0, z(0)=10)(Christropher R Dishop,2019).The infected predator population only available (see Graph 1) is the periodic point at 3.67879350770605. The infected predator population is decreasing due to the absence of a prey population (Vijaya S & Rekha E, 2017; S.P.Bera, Y. A., 2012).
- If we determine the variables as ρ_1 of the structure [31], as mentioned above, the starting condition satisfies with (x(0)=0, E(0)=1, y(0)=0,z(0)=0). The exposed predator population is available (see Graph 2), which is the periodic point at 0.367879356307219 and the susceptible predator population is decreasing due to the absence of a prey population (Vijaya S & Rekha E, 2017; S. P. Bera, Y. A, 2012).
- If we determine the variables as ρ_1 of the structure[31], as mentioned above, the starting condition satisfies with (x(0)=0, E(0)=0, y(0)=1,z(0)=0). The Susceptible predator population is available (see Graph 3), which is the periodic point at 0.367879356307219 and the Exposed predator population is decreasing due to the absence of a prey population (Vijaya S & Rekha E, 2017).
- If we determine the variables as ρ_1 of the structure as mentioned above, the starting condition satisfies with (x(0)=1, E(0)=0, y(0)=0,z(0)=0). The prey population is available (see Graph 4), which is the periodic point at 9.99591738061742 and the prey population is increasing due to the absence of a predator population (Vijaya S & Rekha E, 2017).
- If we determine the variables as ρ_1 of the structure as mentioned above, the starting condition satisfies with (x(0)=0, E(0)=0.2, y(0)=0.2,z(0)=0.5). The Susceptible-Exposed-Infected predator population is available (see Graph 5), and the Susceptible-Infected Predator population is decreasing due to the absence of a prey population (Vijaya S & Rekha E, 2017).
- If we determine the variables as ρ_1 of the structure as mentioned above, the starting condition satisfied[30] with (x(0)=0, E(0)=0.1, y(0)=0.2,z(0)=0). The susceptible-exposed population is decreasing due to the absence of a prey population (see Graph 6) (Vijaya S & Rekha E, 2017).
- Now we take the variables as ρ_1 of the structure as mentioned above, the starting condition satisfies with (x(0)=1.0, E(0)=0, y(0)=0.1,z(0)=0). From Graph 7, we can see that Relationships take place between prey and susceptible predator species (Vijaya S, J. J, 2017). (Kefeng wang& Feiyue Ye, 2015).
- Now we take the variables as ρ_1 of the structure as mentioned above, the starting condition satisfies with (x(0)=1.0, E(0)=0.1,y(0)=0,z(0)=0). From Graph 8, we can see that Relationships take place between prey and exposed predator species (Vijaya S, J. J, 2017; Kefeng wang & Feiyue Ye, 2015)
- Now, we take the variables as ρ_1 of the structure[33] as mentioned above, the starting condition satisfies with (x(0)=0.1, E(0)=0, y(0)=0,z(0)=1).From Graph 9, we can see that Relationships take place between prey and infected predator species (Vijaya S, J. J, 2017). (kefeng wang & Feiyue Ye, 2015)
- Now, we take the variables as ρ_1 of the structure[33] as mentioned above, the starting condition satisfies with (x(0)=0.15, E(0)=0.15, y(0)=0.15,z(0)=0.15). From Graph 10, we can see that Relationships take place between prey and both Susceptible -Exposed -Infected predator species (Vijaya S, J. J, 2017).
- If we determine the variables of the structure as $\rho_1 = (\alpha = 1, 0.8, 0.2, 0, r = 5, a = 1, c_1 = 1, c_2 = 1, c_3 = 5, c_4 = 1, c_5 = 5, c_6 = 1, d_1 = 1, d_2 = 1)$. The starting condition satisfied (x(0)=0.3, E(0)=0.3, y(0)=0.3, z(0)=0.3),(x(0)=1, E(0)=1, y(0)=1, z(0)=1) for both Susceptible-Exposed-Infected predator and prey population[24] of the time series(see Graph 11,12,13,14,15,16,17)[22] (Christropher R Dishop,2019) (Vijaya S & Rekha E,2017).
- If we determine the variables of the structure as $\rho_1 = (\alpha = 0, 0.5, 1, r = 5, a = 1, c_1 = 1, c_2 = 1, c_3 = 5, c_4 = 1, c_5 = 5, c_6 = 1, d_1 = 1, d_2 = 1)$. The starting condition satisfied (x(0)=0.25, E(0)=0.5,

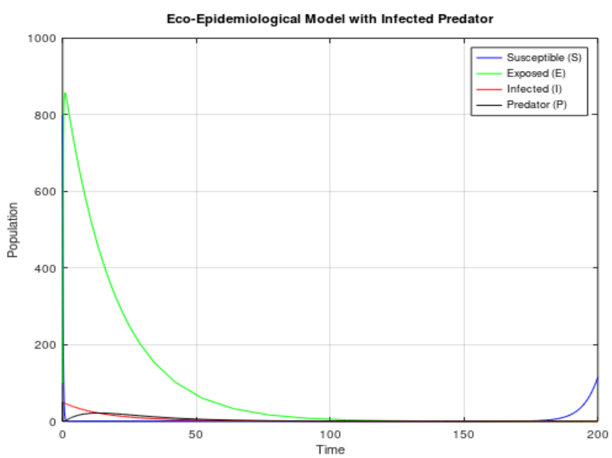


Graph 1: The infected predator populations

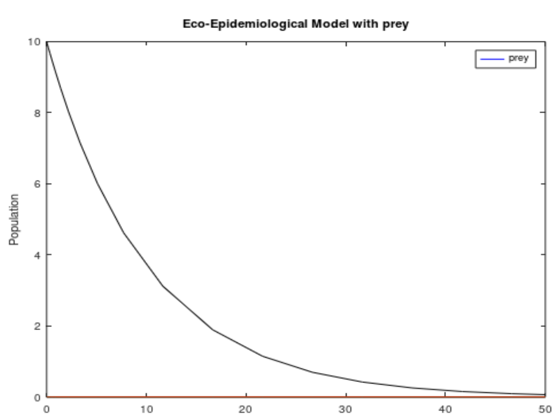
y(0)=0.5, z(0)=0.25),(x(0)=0.5, E(0)=0.5, y(0)=0.5, z(0)=2) ,(x(0)=1, E(0)=0.5, y(0)=0.5, z(0)=1.5) ,(x(0)=1.7, E(0)=0.5, y(0)=0.5, z(0)=1.2) for both Susceptible-Exposed-Infected predator and prey population[24] of the phase plot(see Graph 18,19,20)[22] (Christropher R Dishop,2019) (Vijaya S & Rekha E,2017).



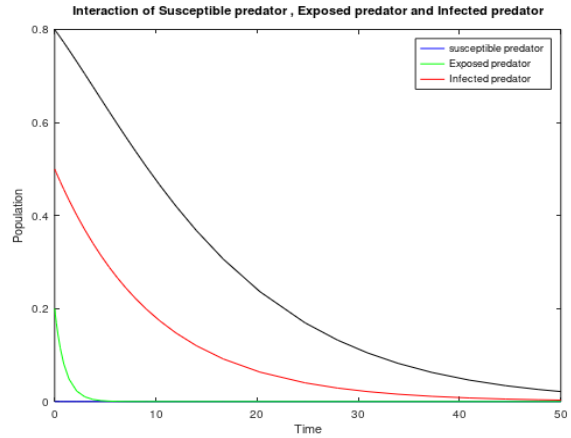
Graph 2: The exposed predator populations



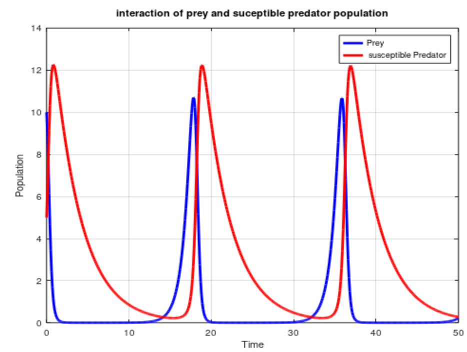
Graph 3: The susceptible predator populations



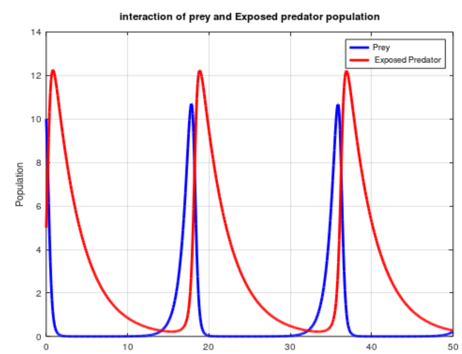
Graph 4: The prey populations



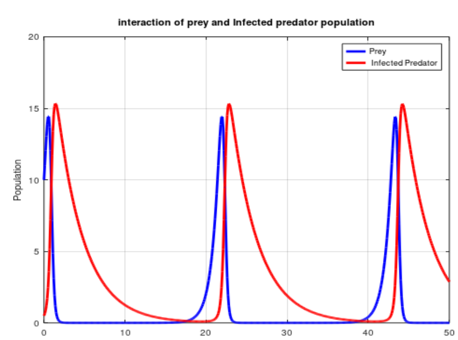
Graph 5: Relationship of susceptible predator exposed predator and infected predator populations



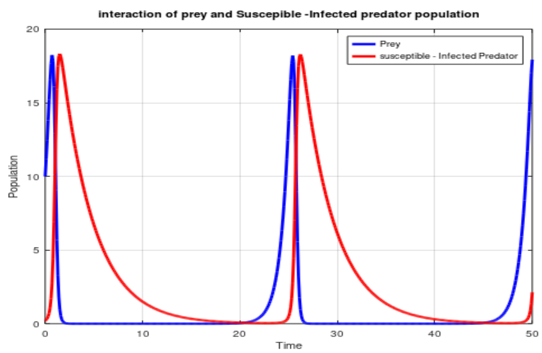
Graph 6: Relationship of prey and susceptible predator populations



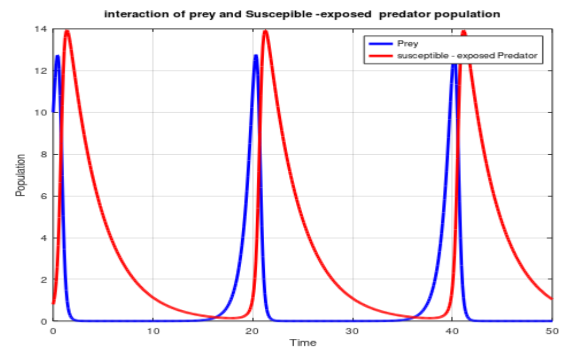
Graph 7: Relationship of prey and exposed predator populations



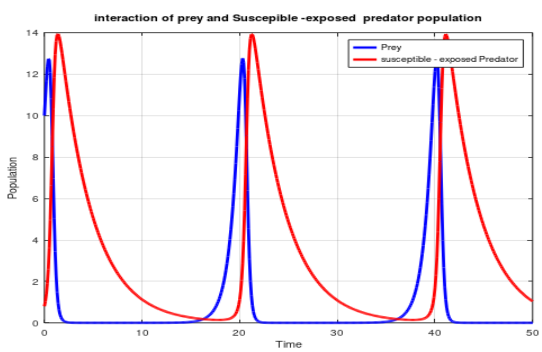
Graph 8: Relationship of prey and infected predator populations



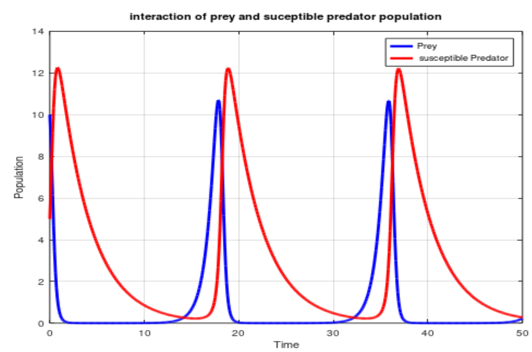
Graph 9: Relationship of prey and susceptible infected predator populations



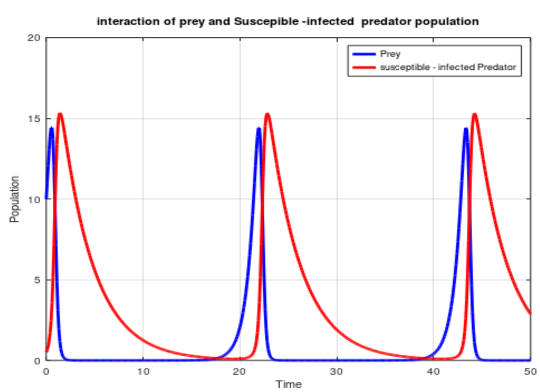
Graph 13: Relationship of prey and susceptible exposed predator populations



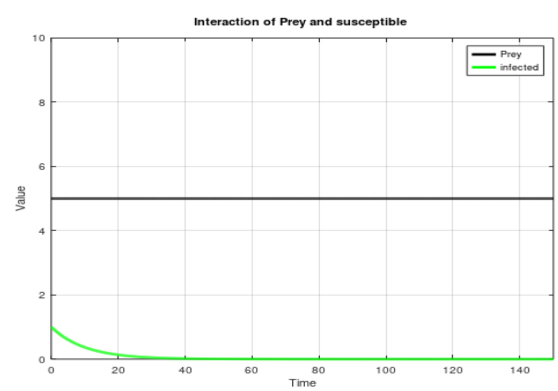
Graph 10: Relationship of prey and susceptible exposed predator populations



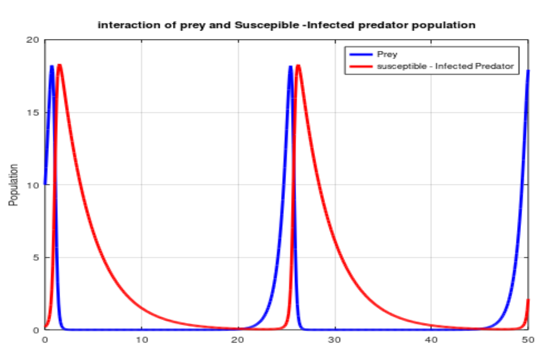
Graph 14: Relationship of prey and susceptible predator populations



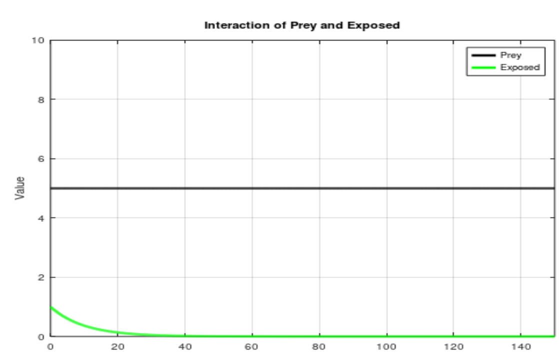
Graph 11: Relationship of prey and susceptible infected predator populations



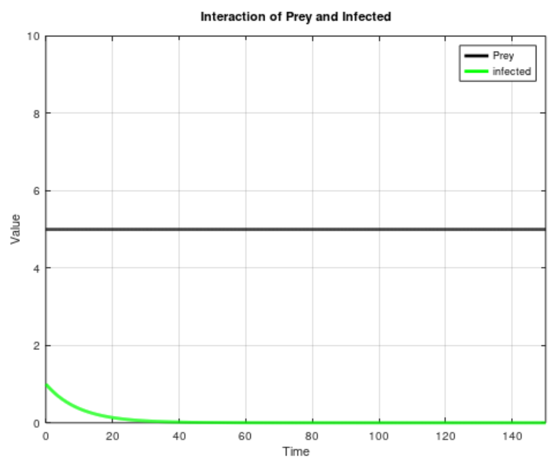
Graph 15: Relationship of prey and infected predator populations



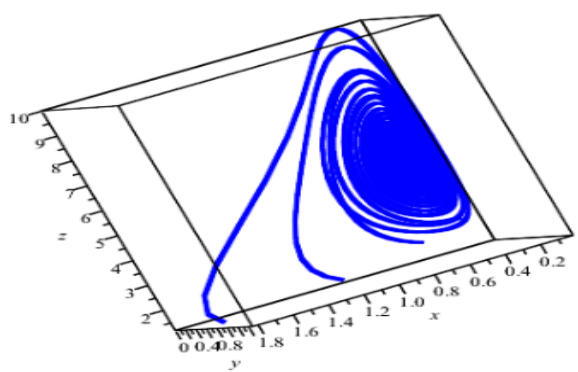
Graph 12: Relationship of prey and susceptible infected predator populations



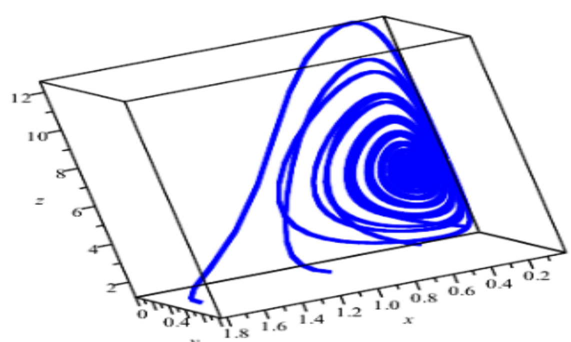
Graph 16: Relationship of prey and exposed predator populations



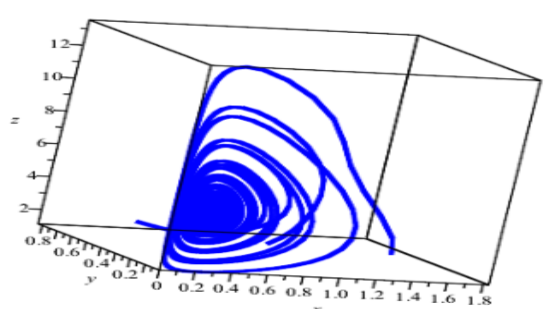
Graph 17: Relationship of prey and susceptible-infected predator populations



Graph 18: Phase plot is asymptotically stable at



Graph 19: Phase plot is asymptotically stable at



Graph 20: phase plot is asymptotically at

Discussion and conclusion

The eco-epidemiological model takes into account the population dynamics of both prey species (represented by susceptible-exposed-infected individuals) and predator species(Vijaya S & Rekha E, 2017). Infection dynamics alter prey dynamics (for example, susceptible individuals become exposed and infected). Predator dynamics are impacted by prey availability, which might fluctuate due to infection-related mortality or behavioral changes in prey species. Prey and predator populations interact in a complex way. Infection in prey can influence predator fitness and population dynamics.

Predators' hunting behavior or energy expenditure may change in response to the incidence of infection in prey species. The eco-epidemiological model, which includes prey species, susceptible-exposed-infected dynamics, and predator species, provides important insights into the dynamics of infectious illnesses in ecological structures. The model improves our knowledge of eco-structure resilience and stability by considering both direct effects (for example, disease-induced mortality) and indirect effects (for example, changes in predator behavior as a result of prey health).

Key results include the ability of disease outbreaks in prey populations to alter predator dynamics, emphasizing the interdependence of species throughout eco-structures. This knowledge is critical for controlling wildlife diseases, conserving biodiversity, and forecasting ecological reactions to environmental change. Additional research could look into complexities like multi-species Relationships and spatial dynamics to improve management tactics and maintain long-term eco-structure health.

In short, eco-epidemiological models provide a powerful framework for investigating the complex links between infectious diseases and animal populations, helping advance ecological theory and practical conservation initiatives. We suggested and analyzed an eco-epidemiological model based on a prey-predator model with SEI sickness in the prey population and susceptible-exposed-infected predators(Subrata Dey, Dhiraj Kumar Das& S. Ghorai, Malay Banerjee,2024). The structure is observed to be positive and limited, with no more than six trivial, disease-free, non-trivial equilibrium positions. Graphs 1, 2, and 4 show that diseased predator populations decrease in the absence of prey while prey populations increase (Graph 3). Graphs 7-14 show the relationship between prey and susceptible, infected predator species.

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