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RESEARCH ARTICLE

Edge properties of lexicographic product graphs of open neighborhood graphs

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Abstract

This research investigates the complex edge characteristics of lexicographic product graphs formed from open neighborhood graphs, filling a notable gap in understanding their structural and adjacency features. Such graphs are pivotal in combinatorial optimization, network architecture, and computational graph theory, particularly for analyzing large-scale systems. By employing rigorous mathematical formulations, the study calculates vertex degrees, edge counts, and degree regularity across diverse graph classes, including cycles, complete graphs, and bipartite structures. A key discovery is the non-commutative nature of the lexicographic product of a graph with its open neighborhood graph, which challenges conventional understandings of graph interactions. MATLAB implementations augment this analysis, providing empirical validation and bridging the gap between theoretical insights and computational applications.

The findings underscore the precision with which edge properties and adjacency relationships can be characterized, offering a harmonious integration of abstract theory with real-world applicability. This research enriches the comprehension of graph dynamics, catering to the needs of scholars and practitioners in computer science, telecommunications, and data analytics. By laying a robust groundwork for future inquiries into graph optimization and network analysis, the study establishes itself as a cornerstone in discrete mathematics. It also highlights the transformative potential of computational tools in elucidating complex network structures.

Keywords: Lexicographic product graphs, Regular graphs, Open neighborhood graphs, Adjacency properties, MATLAB implementations, Degree regularity.

Introduction

Open Neighborhood Graphs represent a fundamental concept in graph theory, derived by analyzing the open neighborhood of a vertex, which includes its adjacent vertices but excludes the vertex itself. Introduced by V.R. Kulli, this class of graphs provides insights into adjacency configurations and connectivity, playing a pivotal role in combinatorial optimization, network design, and domination theory (V.R.Kulli, 2010,2012). The structural complexity and adaptability of open neighborhood graphs facilitate applications across domains such as computer science, operations research, and sociology (Bondy J A. *et al*.., 2008; V.R. Kulli, 2010)

Graph theory, a field that has witnessed exponential growth during the 20th century, underpins modern computational tools and methodologies (Harary.F, 1969). Open neighborhood concepts, formalized within this framework, enable the study of adjacency in both theoretical and applied contexts. For instance, the lexicographic product of two graphs enhances structural analysis by constructing a composite graph, which models layered network interactions effectively (Gert Sabidussi,1961), (Rahman, *et al.*, 2017). This product is critical in understanding the behavior of large-scale and interconnected systems.

The open neighborhood set denoted $N(v)$, for a vertex v in a graph G, forms the basis for constructing open neighborhood graphs. Such representations are not only theoretical constructs but are also integral to practical applications, including social network analysis, communication networks, and biological systems in the Literature (A. Barabási and R. Albert, 1999), (P. Erdos and A. Rényi, 1959). They have been utilized to optimize resource allocation, model traffic flow, and improve connectivity in

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wireless networks according to (C. Berge, 1973), (J. Kleinberg, 2000).

Moreover, the lexicographic product, denoted $G[H]$, where G and H are graphs, extend these concepts by defining a graph whose vertex set is the Cartesian product of G and H. The adjacency conditions in $G[H]$ create opportunities for deeper exploration into graph products, their properties, and computational implementations (M. Newman, 2010). This paper explores these properties and investigates the edge-related characteristics of lexicographic product graphs and their open neighborhood counterparts using MATLAB simulations of the literature (D.B. West, 2001), (L. Lovasz, 1979).

The findings outlined in this study contribute to the growing body of knowledge in graph theory and its applications, particularly emphasizing the non-commutative nature of lexicographic products accordingly (T. Nishizeki and *et al*.., 1988).

Preliminaries

For any Regular finite, undirected and simple graph $G=(V,E)$ where $|V|=p$ and $|E|=q$, the open neighborhood of a vertex $u \in V$ is defined as $N(u)=\{v \in V : uv \in E\}$ denoted as the open neighborhood set $S=\{N(u_1),N(u_2),\ldots..N(u_p)\}$ for all vertices in G by (J. Kleinberg, 2000).

The lexicographic product of two graphs G and H, denoted $G[H]$, constructs a graph whose vertex set is $V(G[H])=V(G)\times V(H)$ with edges defined such that $(u,v),(u',v') \in E(G[H])$◆$u=u'$ and $vv' \in E(H)$ or u is adjacent to $u' \in E(G)$ as (L. Lovasz, 1979). This product enriches the structural analysis of graph interactions, providing tools to model and evaluate multi-layered networks.

In this paper sections 2.2 & 2.3, we investigate certain edge-related properties of lexicographic products of graphs with its open neighborhood Graphs through MATLAB Implementing & two open neighborhood graphs are not commutative.

Main Results

Lexicographic product graph of G and its open neighborhood graph N(G)

Theorem

For cycle $\boldsymbol{C_n}$ for $\boldsymbol{n} \geq 4$, The total number of edges in Lexicographic Product Graph $G=\boldsymbol{C_n}[N(\boldsymbol{C_n})]$ and its open neighborhood graph $N(\boldsymbol{C_n})$ is $2\boldsymbol{n}^2(2\boldsymbol{n}+1)$.

Proof:

Let C_n be a cycle of the length is n. Then it has n vertices & edges are n and the degree of every vertex is 2. i.e., C_n is a 2 regular graph. And let the number of vertices in C_n be represented as $\{1,2,3,4,\ldots\ldots n\}$.

Consider the open neighborhood graph $N(C_n)$ of cycle C_n. Then $N(C_n)$ has a cycle of length $2n$. Where it is also a 2 regular graph.

Then the vertex set of $N(C_n)$ consists of the vertices $1,2,3,\ldots.n,N(1),N(2),\ldots..N(n)$ and therefore $|V(N(C_n))|=2n.$ Now consider the Lexicographic Product of C_n & $N(C_n)$ denoted by $C_n[N(C_n)]=G(say)$ $V(C_n)XV(N(C_n))=\{(1,1),(1,2),\ldots.($, where the vertex set of G is $V(C_n)XV(N(C_n))=\{(1,1),(1,2),\ldots.(,(n,N(n))\}$ and $V(G)=V(C_n)XV(N(C_n))=\{(u_i,v_j);u_i \in V(C_n),v_j \in V(N(C_n))\}$ where $1 \leq i \leq n, 1 \leq j \leq 2n$.

$\therefore$ The total number of vertices in G are üü = ^{2} and G is also 2 regular graph.

In Lexicographic Product, any two vertices $(u_i,v_j),(u_{i'},v_{j'})$ are said to be adjacent in G.

If (i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(C_n)$ (or) (ii) u_i is adjacent to $u_{i'}$ in C_n.

Let (u_i,v_j) ,$(u_{i'},v_{j'})$ be any two vertices in G. For adjacency of vertex in G and to find the degree of each vertex in G.

Case (I): *consider the condition (i),*

i.e., $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(C_n)$. Each vertex in G contains two edges, which are adjacent in $N(C_n)$.Therefore, degree of each vertex is 2. i.e., $d(u_i,v_j)=2.$

Case (II): For C_n, every vertex üu_i v_i $\in$ C_n has two neighboring vertices. Each vertex $(u_i,v_j) \in G$ s connected to all vertices $(u_{i'},v_{j'})$, where $u_{i'}$ is a neighbor of u_i in C_n.For each $u_{i'}$ there exist $2n$ adjacent vertices v_j in $N(C_n)$.

Since there are $2n$ vertices are distinct in the G and hence each vertex has degree 2.

Therefore, $2(2n)$ vertices are adjacent in G. Then the degree of each vertex in G are $d(u_i,v_j)=4n$

Combining the above conditions (i) & (ii), the degree of each vertex in G is ü$n+$

In the graph G the degree of each vertex (u_i,v_j) is $d(u_i,v_j)=$ ü$n+$

But in the graph G (Product of $C_n[N(C_n)]$) then all vertices are $2n^2$.

So, the all vertices of degree in G are $2n^2 \times d(u_i,v_j)=2n^2(4n+2)$.

Hence the total degree in G is $\sum_{i=1}^{n} d(u_i,v_j)=2n^2(4n+2)$.

Let $\in$ denotes the total number of edges in G.By Handshaking Theorem,

$$\sum_{i=1}^{n} d(u_i,v_j) \quad = 2\in◆4n^2(2n+1) \quad = 2\in◆ \quad \in = 2n^2(2n+1)$$

Hence the all edges are in G is $2n^2(2n+1)$.

Example: If $n=4$, (Figures 1 and 2)

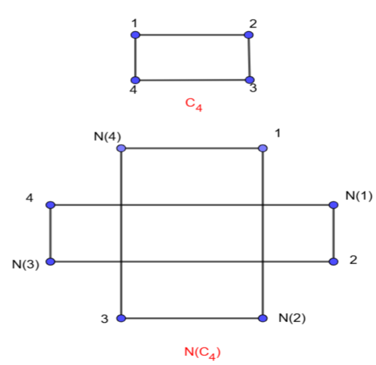

Figure 1. Cycle graph C_4 & it's open neighborhood graph $N(C_4)$

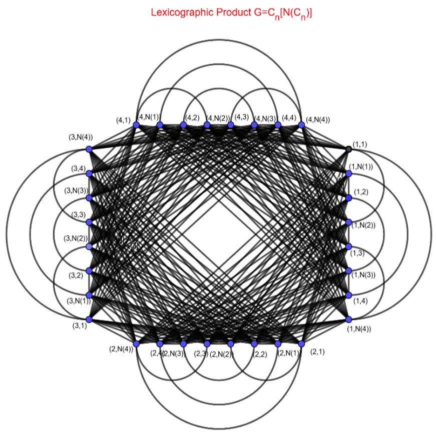

Figure 2: Lexicographic product of $C_4\left[N(C_4)\right]$

Matlab Code

```
%MATLAB script to calculate the total number of vertices, degrees,
%and edges in G = C_n[N(C_n)] for a given value of n (n >= 4)
%Prompt the user to input the value of n
n = input('Enter the value of n (n >= 4): ');
%Validate the input
if n < 4
error('The value of n must be greater than or equal to 4.');
end
%Step 1: Compute the total number of vertices in G
num_vertices = 2 * n^2;
%Step 2: Compute the degree of each vertex in G
degree_per_vertex = 4 * n + 2;
%Step 3: Compute the total degree of all vertices in G
total_degree = num_vertices * degree_per_vertex;
%Step 4: Use the Handshaking Theorem to calculate the total number of edges
total_edges = total_degree / 2;
%Display the results
Fprintf ('The number of vertices in G = C_%d[N(C_%d)] is: %d\n', n, n, num_vertices);
fprintf('The degree per vertex in G = C_%d[N(C_%d)] is: %d\n', n, n, degree_per_vertex);
fprintf('The total degree of all vertices in G = C_%d[N(C_%d)] is: %d\n', n, n, total_degree);
fprintf('The total number of edges in G = C_%d[N(C_%d)] is: %d\n', n, n, total_edges);
```

Output

Enter the value of n (n >= 4): 4
The number of vertices in G = C_4[N(C_4)] is: 32
The degree per vertex in G = C_4[N(C_4)] is: 18
The total degree of all vertices in G = C_4[N(C_4)] is: 576
The total number of edges in G = C_4[N(C_4)] is: 288

Corollary

The total number of edges in the Lexicographic Product Graph $G = N(C_n)[C_n]$, where C_n is a cycle graph with $n \geq 4$, along with its open neighborhood graph $N(C_n)$, is expressed as ün^2 $n+$).

Proof:

Let $N(C_n)$ be the open neighborhood graph of cycle C_n. Then it has cycle of length $2n$. Which is a 2 regular graph.

Let $V\left(N(C_n)\right)=\{1,2,3,\ldots n,N(1),N(2),\ldots\ldots N(n)\}$ and $|V(N(C_n))| = 2n$ and $V(C_n)=\{1,2,3,4,\ldots\ldots n\}$ vertex set of C_n such that $|V(C_n)| = n$.

Now consider the Lexicographic Product of $N(C_n)\,\&C_n$ denoted by $C_n\left[N(C_n)\right] = G(say)$, where the vertex set of G is

$V\left(N(C_n)\right)X\,V(C_n) = \{(1,1),(2,1),\ldots(n,1),(N(1),1),(N(2),1),.$

$..,(N(n),1),(1,2),(2,2)\ldots,(n,2),(N(1),2)\ldots(N(n),2)$

$,\ldots,(N(n),2),\ldots.,(1,n),(2,n)\ldots,(n,n),(N(1),n)\ldots(N(n),n)\}$.

$V(G)=V\left(N(C_n)\right)X\,V(C_n)=\{(u_i,v_j)\,;u_i\in V(N(C_n)),v_j\in V(C_n)\}$,

where $1\leq i\leq 2n, \quad 1\leq j\leq n$.

Therefore, the total number of vertices in G are $2n\times n = 2n^2$.

In Lexicographic Product, any two vertices $(u_i,v_j),(u_{i'},v_{j'})$ are said to be adjacent in G.

If (i) $u_i = u_{i'}$ and v_j is adjacent to $v_{j'}$ in C_n (or) (ii) u_i is adjacent to $u_{i'}$ in $N(C_n)$.From the above theorem, we say that in the Lexicographic Product $C_n\left[N(C_n)\right]$ any two vertices are adjacent.

From case(i), we have the degree of each vertex is 2. And case from (ii), since there are $2n$ distinct vertices from $N(C_n)$ are adjacent to each vertex in C_n.

$\therefore$ The degree of each vertex in this case is $2n$. Therefore, every vertex of degree in G are ü$n+$

So, the total degree of all vertices in G is

$$2n^2\times d(u_i,v_i) = 2n^2\times 2(n+1) = 4n^2(n+1)$$

Hence the total degree in G is $\sum_{i=1}^{n} d(u_i,v_j) = 4n^2(n+1)$

Let $\in$ be the total number of edges in G and Using the handshaking theorem, we get,

$$\sum_{i=1}^{n} d(u_i,v_j) = 2\in\ 4n^2(n+1) = 2\in\ \in = 2n^2(n+1)$$

Hence the total number of edges are in G is $2n^2(n+1)$.
Hence theorem is proved.

Matlab Code

```
%MATLAB script to calculate the total number of vertices, degrees,
%and edges in G = N(C_n)[ C_n] for a given value of n (n >= 4)
%Validate the input
if n < 4
error ('The value of n must be greater than or equal to 4');
end
%Step 1: Calculate the total number of vertices in G
total_vertices = 2 * n^2; %Total vertices in G = 2n^2
```

fprintf('The total number of vertices in G = N(C_%d)[C_%d] is: %d\n', n, n, total_vertices);
%Step 2: Calculate the degree of each vertex in G
degree_per_vertex = 2 * n + 2; %Degree of each vertex = 2n + 2
fprintf('The degree of each vertex in G = N(C_%d)[C_%d] is: %d\n', n, n, degree_per_vertex);
%Step 3: Calculate the total degree of all vertices in G
total_degree = total_vertices * degree_per_vertex; %Total degree = Sum of vertex degrees
fprintf('The total degree of all vertices in G = N(C_%d)[C_%d] is: %d\n', n, n, total_degree);
%Step 4: Calculate the total number of edges in G using the Handshaking Theorem
total_edges = total_degree / 2; %Total edges = Total degree / 2
fprintf('The total number of edges in G = N(C_%d)[C_%d] is: %d\n', n, n, total_edges);

Output

Enter the value of n (n >= 4): 4
The total number of vertices in G = N(C_4)[C_4] is: 32
The degree of each vertex in G = N(C_4)[C_4] is: 10
The total degree of all vertices in G = N(C_4)[C_4] is: 320
The total number of edges in G = N(C_4)[C_4] is: 160
(Figure 3)

$(n,1),(n,2),\ldots\ldots,(n,n),(N(n),N(1)),\ldots.,(N(n),N(n))\}$

Theorem

Let $G=K_n[N(K_n)]$ be the Lexicographic product graph of the Complete Graph $K_n (n\geq 3)$ and its Open Neighborhood graph $N(K_n)$. Then the total number of edges is $n^2(2n+1)(n-1)$.

Proof

Let K_n be a Complete Graph with n vertices & n edges and every vertex of degree is $(n-1)$. Then K_n is a $(n-1)(=n_{C_2})$ regular graph and the set $\{1,2,3,4,\ldots\ldots n\}$ denotes the vertices of K_n.

Consider the Open Neighborhood Complete Graph $N(K_n)$ of the Complete Graph K_n. Then $N(K_n)$ is a $2n$ vertices and it is also a $(n-1)$ regular graph.

Then the vertex set of $N(K_n)$ is $\{1,2,3,\ldots.n,N(1),N(2),\ldots..N(n)\}$and such that $|V(N(K_n))|=2n.$

Now consider the Lexicographic Product of K_n & $N(K_n)$ denoted by, where the vertex set of G is

$$V(K_n)XV(N(K_n))=(1,1),(1,2),\ldots.(1,N(1)),(1,N(2)).,,,,$$
$$(1,N(n)),(2,1),(2,2)\ldots.(2,n),(2,N(1)).,,,,(2,N(n)),\ldots,$$

In general,

$$V(G)=V(K_n)XV(N(K_n))=\{(u_i,v_j)\ ;u_i\in V(K_n),v_j\in V(N(K_n))\},$$

where $1\leq i\leq n, 1\leq j\leq 2n$.

Therefore, the total number of vertices in G are üẍ = ² and G is also $(n-1)$ regular graph.

In Lexicographic Product, any two vertices $(u_i,v_j),(u_{i'},v_{j'})$ are said to be adjacent in G. If

(i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(K_n)$ (or) (ii) u_i is adjacent to $u_{i'}$ in K_n.

Let (u_i,v_j) ,$(u_{i'},v_{j'})$ be any two vertices in G.

For adjacency of vertices in G and to find the degree of each vertex in G. we have the following two cases:

Case (I): consider the condition (i), for adjacency i.e., $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(K_n)$. � Each vertex in G contains two edges, which are adjacent in $N(K_n)$. Therefore, degree of each vertex is $(n-1)$.

$$\text{i.e., } d(u_i,v_j)=(n-1).$$

Case (II): For C_n,each vertex üu_i v_j $\in$ K_n has $(n-1)$ neighbours vertices in K_n and hence the vertex $(u_i,v_j)\in G$ is adjacent to $(n-1)$ vertices $(u_{i'},v_{j'})$. (Where $u_{i'}$ is adjacent to u_i in K_n and v_j is adjacent to $v_{j'}$ in $N(K_n)$).

i.e., For each $u_{i'}$, there are $2n(n-1)$ adjacent vertices $v_{j'}$ in $N(K_n)$.

Since $2n(n-1)$ vertices are distinct in G and hence each vertex has degree $2n(n-1)$.Therefore, $2n(n-1)$ vertices are adjacent in G. Then the degree of each vertex in G. i.e., $d(u_i,v_j)=2n(n-1)$

Combining the above conditions (i), (ii), we have, the degree of each vertex in G is

$$d(u_i,v_j)=(n-1)+2n(n-1)=(2n+1)(n-1)$$

But in the graph G (= $K_n[N(K_n)]$) the total number of vertices are $2n^2$. So, the total degree of all vertices in G is $2n^2\times d(u_i,v_j)=2n^2(2n+1)(n-1)$

Hence the total degree in G is $\sum_{i=1}^{n} d(u_i,v_j)=2n^2(2n+1)(n-1)$

Let $\in$ be the total number of edges in G.

By Handshaking Theorem, $\sum_{i=1}^{n} d(u_i,v_j)=2\in$ �$2n^2(2n+1)(n-1)=2\in$

$$\therefore \quad \in=n^2(2n+1)(n-1)$$

$\therefore$ The total number of edges in G are $n^2(2n+1)(n-1)$.

Ex: If $n=3$, (Figures 4 and 5).

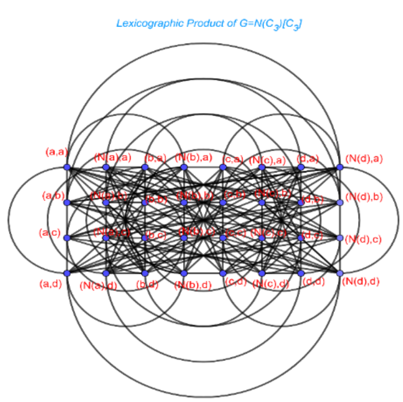

Figure 3: Lexicographic product of $N(C_4)[C_4]$

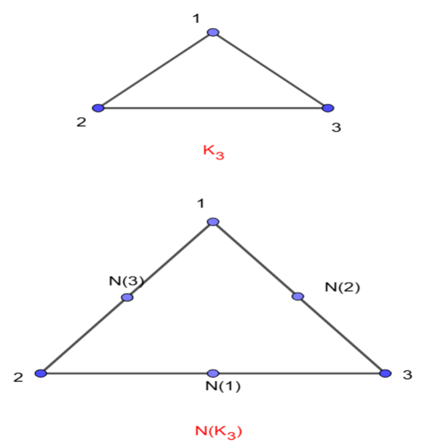

Figure 4: Complete Graph K_3 & it's open neighborhood graph $N(K_3)$

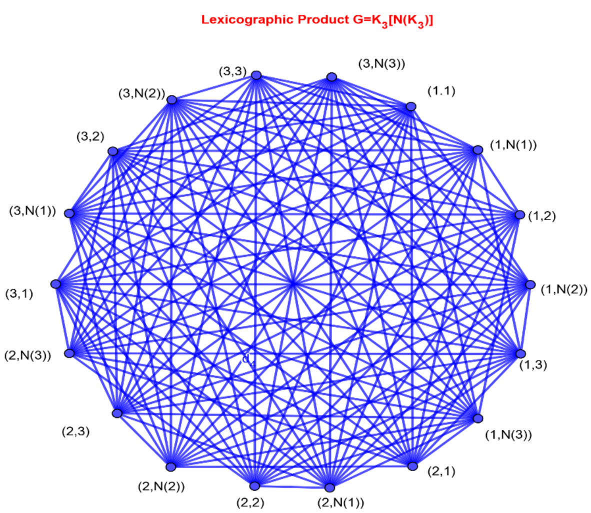

Figure 5: Lexicographic Product of $K_3[N(K_3)]$

MATLAB CODE

```
%Function definition
function totalEdges = lexicographicEdgesGeneral()
%Prompt the user to input a value for n
n = input('Enter the value of n (n >= 3): ');
 %Step 1: Validate the input
if n < 3
error('n must be greater than or equal to 3.');
end
%Step 2: Calculate the number of vertices in G
vertices_In_NKn = 2 * n;            %N(K_n) has 2n vertices
vertices_In_G = n * vertices_In_NKn;     %Total vertices in G = n * 2n
 %Step 3: Calculate the degree of each vertex and total degree
degree_Each_Vertex = (n - 1) + 2 * n * (n - 1); %(n-1) + 2n(n-1)
total_Degree = vertices_In_G * degree_Each_Vertex; %Total degree
 %Step 4: Calculate total edges using both methods
total_Edges = total_Degree / 2;   %Total edges by Handshaking Theorem
direct_Calculation = n^2 * (2 * n + 1) * (n - 1); %Formula result
 %Display results
fprintf('For n = %d:\n', n);
fprintf ('Number of vertices in G: %d\n', vertices_In_G);
fprintf ('Degree of every vertex in G: %d\n', degree_Each_Vertex);
fprintf ('Total degree of all vertices in G: %d\n', total_Degree);
fprintf ('Total number of edges (calculated) in G: %d\n', total_Edges);
end
```

OUTPUT

Enter the value of n (n >= 3): 3
For n = 3:
Number of vertices in G: 18
Degree of every vertex in G: 14
Total degree of all vertices in G: 252
Total number of edges (calculated) in G: 126

Corollary

The Lexicographic product $\boldsymbol{G}=\boldsymbol{N}(\boldsymbol{K_n})[\boldsymbol{K_n}]$ of the Complete Graph $\boldsymbol{K_n}(\boldsymbol{n}\geq 3)$ and its open neighborhood graph $N(\boldsymbol{K_n})$. Then the total number of edges in G is calculated as $\boldsymbol{n}^2(\boldsymbol{n}^2-1)$.

Proof

Let the open neighborhood graph $N(K_n)$ of complete graph K_n. Then the degree of vertices K_n is $2n$. Where it is also a $(n-1)$ regular graph. The vertex set of $N(K_n)$ consists of the vertices $1,2,3,....n,N(1),N(2),.....N(n)$ and therefore $|V(N(K_n))|=2n.$

Consider K_n be a complete graph has n vertices & n edges and the degree of each vertex is $(n-1)$. i.e., K_n is also $(n-1)$ regular graph.

Let $\{1,2,3,4,......n\}$ denotes the number of vertices in K_n.

Now consider the lexicographic product of $N(K_n)\,\&K_n$ denoted by $N(K_n)[K_n]=G(say)$, where the vertex set of G is $V(N(K_n))XV(K_n)=\{(1,1),(2,1),....(N(1),1),(N(2),1).,,,$ $,(N(n),1),(1,2),(2,2)....(n,2),(N(1),2).,,,,(N(n),2),....,(N(n),n)\}$ $.V(G)=V(N(K_n))XV(K_n)=\{(u_i,v_j)\,;u_i\in V(N(K_n)),v_j\in V(K_n)\}$

where $1\leq i\leq 2n, 1\leq j\leq n$.

Therefore, the total number of vertices in G is $2n\times n=2n^2$

In Lexicographic Product, if $(u_i,v_j),(u_{i'},v_{j'})$ be any two vertices are said to be adjacent in G.

If (i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in K_n (or) (ii) u_i is adjacent to $u_{i'}$ in $N(K_n)$.From the above theorem, we say that in the Lexicographic Product $N(K_n)[K_n]$ any two vertices are adjacent.

From case(i), we have, The degree of each vertex is $(n-1)$. And case from (ii), since there are $n(n-1)$ distinct vertices from $N(C_n)$ are adjacent to each vertex in C_n.

$\therefore$ The degree of each vertex in this case is $n(n-1)$.

Therefore, the degree of each vertex in G is

$d(u_i,v_i)=n(n-1)+(n-1)=(n-1)(n+1)=n^2-1$

But in the G graph (Product of $N(K_n)[K_n]$) the total number of vertices are $2n^2$.

So, the total degree of all vertices in G is

$2n^2\times d(u_i,v_i)=2n^2\times n^2-1=2n^2(n^2-1)$

Hence the total degree in G is $\sum_{i=1}^{n} d(u_i,v_i)=2n^2(n^2-1)$

Let $\in$ be the total number of edges in G.

By Handshaking Theorem, $\sum_{i=1}^{n} d(u_i,v_i) = 2\in$ �$2n^2(n^2-1)=2\in$

$\therefore \in=n^2(n^2-1)$

Hence the total number of edges are in G is $n^2(n^2-1)$.

MATLAB CODE

```
%MATLAB Code for the Lexicographic Product G = K_n[N(K_n)]
%The total number of edges in G is calculated as n^2 * (n^2 - 1)
 %Input: Prompt the user to enter the value of n
n = input('Enter the value of n (n >= 3): ');
 %Step 1: Validate the input
if n < 3
error('n must be greater than or equal to 3.');
end
%Step 2: Calculate the number of vertices in G
verticesInNKn = 2 * n; %N(K_n) has 2n vertices
verticesInG = n * verticesInNKn; %Total vertices in G = n * 2n = 2n^2
%Step 3: Calculate the degree of each vertex in G
degreeEachVertex = n^2 - 1; %Degree = (n-1)(n-1) + n(n-1)
%Step 4: Calculate the total degree and total number of edges in G
totalDegree = verticesInG * degreeEachVertex; %Total degree = vertices * degree per vertex
totalEdges = totalDegree / 2; %Total edges = Total degree /
%Display the results in the desired format
fprintf('Enter the value of n (n >= 3): %d\n', n);
fprintf('The total number of vertices in G = K_%d[N(K_%d)] is: %d\n', n, n, verticesInG);
fprintf('The degree of each vertex in G = K_%d[N(K_%d)] is: %d\n', n, n, degreeEachVertex);
fprintf('The total degree of all vertices in G = K_%d[N(K_%d)] is: %d\n', n, n, totalDegree);
fprintf('The total number of edges in G = K_%d[N(K_%d)] is: %d\n', n, n, totalEdges);
```

OUTPUT

```
Enter the value of n (n >= 3): 3
Enter the value of n (n >= 3): 3
The total number of vertices in G = K_3[N(K_3)] is: 18
The degree of each vertex in G = K_3[N(K_3)] is: 8
The total degree of all vertices in G = K_3[N(K_3)] is: 144
The total number of edges in G = K_3[N(K_3)] is: 72
```

Theorem:

The total number of edges in Lexicographic product $G=K_{n,n}[N(K_{n,n})]$ of Complete Bipartite Graph $K_{n,n}(n\geq 2)$ and its open neighborhood graph $N(K_{n,n})$ is $4n^3(4n+1)$.

Proof

Let $K_{n,n}$ be a Complete Bipartite Graph. Then it has $2n$ vertices & $2n$ edges and the degree of each vertex is n. i.e., $K_{n,n}$ is a n regular graph. Then the set $\{a,b,c,d\ldots\ldots n,1,2,3,4,\ldots\ldots n\}$ is defines the number of vertices in $K_{n,n}$.

Consider the open neighborhood graph $N(K_{n,n})$ of Complete Bipartite Graph $K_{n,n}$. Then $N(K_{n,n})$ has $2(2n)$ vertices. Where it is also a n regular graph.

Then the vertex set of $N(K_{n,n})$ consists of the vertices $a,b,c,\ldots n,1,2,3,\ldots n,N(a),N(b),N(C),\ldots,N(n),N(1),N(2),\ldots..N(n)$ and therefore $|V(N(K_n))|=2(2n)$.

Now consider the Lexicographic Product of $K_{n,n}$ & $N(K_{n,n})$ denoted by $K_{n,n}[N(K_{n,n})]=G(say)$, where the vertex set of G is

$$V(K_{n,n})XV(N(K_{n,n})=\{(a,1),(a,2),\ldots(a,n),(a,N(1)),(a,N(2))\ldots\ldots,(a,N(n)),(b,1),(b,2)\ldots.(b,n),(b,N(1)).,,,,(b,N(n))\ldots,(n,N(1)),(n,N(2)),\ldots..,(n,N(n))\}.$$

$$V(G)=V(K_{n,n})XV(N(K_{n,n}))=\{(u_i,v_j);u_i\in V(K_{n,n}),v_j\in V(N(K_{n,n}))\}$$

where $1\leq i\leq n, 1\leq j\leq 2n$.

Therefore, the total number of vertices in G are $\ddot{u}\ddot{x} \quad = \quad ^2$ and G is also $(n-1)$ regular graph.

In Lexicographic Product, any two vertices $(u_i,v_j),(u_{i'},v_{j'})$ are said to be adjacent in G.

If (i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(K_{n,n})$ (or) (ii) u_i is adjacent to $u_{i'}$ in $K_{n,n}$.

Let $(u_i,v_j),(u_{i'},v_{j'})$ be any two vertices in G.

For adjacency of vertex in G and to find the degree of each vertex in G.

Case (I): consider the condition (i),

i.e., $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $N(K_{n,n})$.

Each vertex in G contains n edges, which are adjacent in $N(K_{n,n})$.

Therefore, degree of each vertex is $(n-1)$. i.e., $d(u_i,v_j)=(n-1)$.

Case (II): For $K_{n,n}$, each vertex $(u_i,v_{j)}\in K_{n,n}$ has $2n$ neighbours.

Each vertex $(u_i,v_j)\in G$ is adjacent to all vertices $(u_{i'},v_{j'})$, where $u_{i'}$ is adjacent to u_i in $K_{n,n}$.

i.e., For each $u_{i'}$, there are $2n(2n)$ adjacent vertices $v_{j'}$ in $N(K_{n,n})$.

Since $2n(2n)$ vertices are distinct in the G and hence each vertex has degree n.Therefore, $2n(2n)$ vertices are adjacent in G.Then the degree of each vertex in G are $d(u_i,v_j)=2n(2n)=4n^2$

Combining the above conditions (i) & (ii),

the degree of each vertex in G is $d(u_i,v_j) = 4n^2 + n$

i.e., In graph G the degree of each vertex (u_i,v_j) is $d(u_i,v_j) = 4n^2 + n$

But in the graph G (Product of $K_{n,n}[N(K_{n,n})]$) the total number of vertices are $(2n)2(2n) = 8n^2$.

So, the total degree of all vertices in G is

$$8n^2 \times d(u_i,v_j) = 8n^2(4n^2+n) = 8n^3(4n+1)$$

Hence the total degree in G is $\sum_{i=1}^{n} d(u_i,v_j) = 8n^3(4n+1)$

Let $\in$ represents the total number of edges in G.
By Handshaking Theorem,

$$\sum_{i=1}^{n} d(u_i,v_j) = 2\in \text{�} 8n^3(4n+1) = 2\in \text{�} \in = 4n^3(4n+1)$$

Hence The Number of edges G is $4n^3(4n+1)$.

Ex: If $n = 2$, (Figures 6 and 7)

MATLAB CODE

```
%MATLAB Code to calculate properties of the graph based on the theorem
clc; clear;
%Step 1: Input for the complete bipartite graph K_{n,n}
n = input(‘Enter the value of n (n >= 2): ‘);
%Step 2: Calculate the number of vertices in the Lexicographic product graph G
vertices_G = 2 * n * (2 * n); %Total vertices in G
disp([‘For n = ‘, num2str(n), ‘:’]);
disp([‘Total vertices in G: ‘, num2str(vertices_G)]);
%Step 3: Calculate the degree of each vertex in G
degree_vertex_G = 4 * n^2 + n; %From the proof
disp([‘Degree of each vertex in G: ‘, num2str(degree_vertex_G)]);
%Step 4: Calculate the total number of edges in G (using simplified formula)
total_edges_G_formula = 4 * n^3 *(4*n+ 1); %Simplified formula from the theorem
disp([‘Total edges in G : ‘, num2str(total_edges_G_formula)]);
```

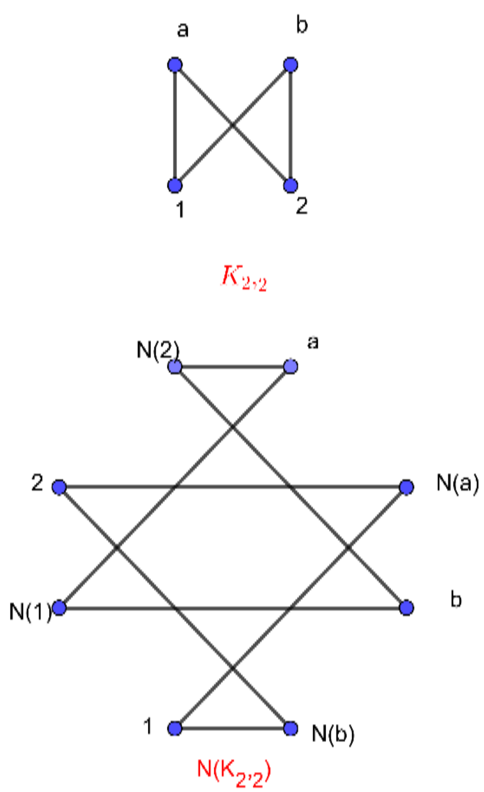

Figure 6: Complete Bipartite Graph $K_{ü}$ & it's open neighborhood graph $N(K_{2,2})$

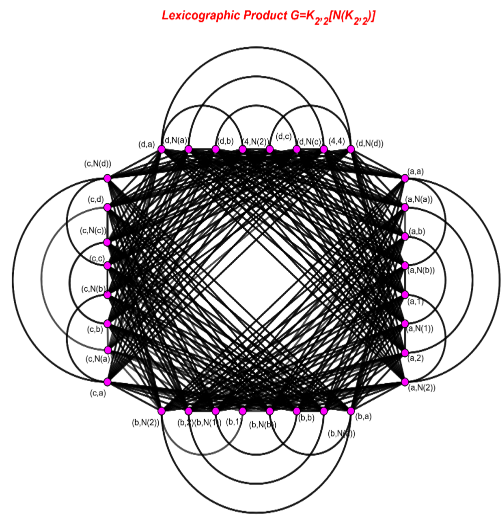

Figure 7: Lexicographic Product of $K_{2,2}[N(K_{2,2})]$

OUTPUT

Enter the value of n (n >= 2): 3
For n = 3:
Total vertices in G: 36
Degree of each vertex in G: 39
Total edges in G : 1404

Corollary

The total number of edges in Lexicographic product $\boldsymbol{G = N(K_{n,n})[K_{n,n}]}$ of Complete Bipartite Graph $\boldsymbol{K_{n,n}}$ $(\boldsymbol{n} \geq 2)$ and its open neighborhood graph $\boldsymbol{N(K_{n,n})}$ is $4\boldsymbol{n}^3(2\boldsymbol{n}+1)$.

Proof

Let the open neighborhood graph $N(K_{n,n})$ of Complete Bipartite Graph $K_{n,n}$. Then it has Complete Bipartite Graph of length $2(2n)$. Where it is also a n regular graph.

Then the vertex set of $N(K_{n,n})$ consists of the vertices

$$\{a,b,c,\ldots.n,1,2,3,\ldots.n,N(a),N(b),N(C),\ldots.,N(n),N(1),N(2),\ldots..N(n)\}$$

and therefore $|V(N(K_{n,n}))| = 2(2n)$

Consider $K_{n,n}$ be a Complete Bipartite Graph. Then it has $2n$ vertices & $2n$ edges and the degree of each vertex is n. i.e., $K_{n,n}$ is also n regular graph.

Let $\{a,b,c,\ldots\ldots n,1,2,3,4,\ldots\ldots n\}$ be represents the number of vertices in $K_{m,n}$.

Now consider the Lexicographic Product of $N(K_{n,n})$ & $K_{n,n}$ denoted by $N(K_{n,n})[K_{n,n}]=G(say)$, where the vertex set of G is

$V(N(K_{n,n}))XV(K_{n,n})=\{(1,a),(2,a),\ldots,(n,a),(N(1),a),(N(2),a),\ldots.,(N(n),a),$

$(1,b),(2,b)\ldots(n,b),(N(1),b),\ldots,(N(n),b),\ldots,(1,n),,(n,n),(N(1),n),,\ldots..,(N(n),n)\}$

. $V(G)=V(N(K_{n,n}))XV(K_{n,n})=\{(u_i,v_j);u_i\in V(N(K_{n,n})),v_j\in V(K_{n,n})\}$,

where $1\le i\le 2(2n),1\le j\le 2n$.

Therefore, the total number of vertices in G are $2(2n)\times 2n=8n^2$ and G is also n regular graph.

In Lexicographic Product Graph G, any two vertices $(u_i,v_j),(u_{i'},v_{j'})$ are said to be adjacent in G.

If (i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in $K_{n,n}$ (or) (ii) u_i is adjacent to $u_{i'}$ in $N(K_{n,n})$. From the above theorem, we say that in the Lexicographic Product $N(K_{n,n})[K_{n,n}]$ any two vertices are adjacent.

From case(i), we have, the degree of each vertex is n.

And case from (ii), since there are $n(2n)$ distinct vertices from $N(K_{n,n})$ are adjacent to each vertex in $K_{n,n}$.

$\therefore$ The degree of each vertex in this case is $n(2n)=2n^2$.

Therefore, the degree of each vertex in G is $d(u_i,v_j)=2n^2+n$

But in the graph G (Product of $N(K_{n,n})[K_{n,n}]$) the total number of vertices are $(2n)(2)(2n)=8n^2$.

So, the total degree of all vertices in G is

$$8n^2\times d(u_i,v_j)=8n^2\times(2n^2+n)=8n^2(2n^2+n)$$

Hence the total degree in G is $\sum_{i=1}^{n}d(u_i,v_j)=8n^2(2n^2+n)$

Let $\in$ be the total number of edges in G.

By Handshaking Theorem,

$$\sum_{i=1}^{n}d(u_i,v_j)=2\in 8n^2(2n^2+n)=2\in \quad \in=4n^3(2n+1)$$

Hence all edges are in G is $4n^3(2n+1)$

MATLAB Code

```
%MATLAB Code to calculate the total number of edges in Lexicographic Product Graph
%Theorem: Total edges in G = N(K_{n,n})[K_{n,n}] is 4n^3(2n + 1)
%Parameters
n = input('Enter the value of n (n >= 2): ');%User-defined value for n
%Calculation
%Number of vertices in G
num_vertices = 2 * (2 * n) * (2 * n); % = 8n^2
%Degree of each vertex in G
vertex_degree = 2 * n^2 + n;
%Total degree of all vertices
total_degree = num_vertices * vertex_degree;
%By Handshaking Theorem: Total edges in G
total_edges = total_degree / 2;
%Simplified formula for total edges
simplified_edges = 4 * n^3 * (2 * n + 1);
%Display results
fprintf('For n = %d:\n', n);
fprintf('Total vertices in G : %d\n', num_vertices);
fprintf('Degree of each vertex in G : %d\n', vertex_degree);
fprintf('Total edges in G (calculated): %d\n', total_edges);
fprintf('Total edges in G (simplified formula): %d\n', simplified_edges);
%Verify the nth place value
nth_place_value = 4 * n^3 * (2 * n + 1);
fprintf('The value at nth place is: %d\n', nth_place_value);
```

OUTPUT

Enter the value of n (n >= 2): 2
For n = 2:
Total vertices in G : 32
Degree of each vertex in G : 10
Total edges in G (calculated): 160
Total edges in G (simplified formula): 160
The value at nth place is: 160

NOTE 1: The lexicographic product of a Graphs with its Open Neighborhood Graphs is not Commutative.

The Lexicographic Product Graph of two open neighborhood graphs

Theorem

The total number of edges in Lexicographic product Graph $G=G_1[G_2]$ of two Regular open neighborhood graphs G_1 & G_2 is $2mn(2nk_1+k_2)$ for $m\ge 3,n\ge 4,k_1,k_2\ge 2$.

Proof

Let G_1 be a k_1 regular Open Neighborhood Graph. Then it has $2m$ vertices.

Now $\{a,b,c,\ldots\ldots,m,N(a),N(b),N(c),\ldots,N(m)\}$ be the vertices in G_1.

Let G_2 be another k_2 regular Open Neighborhood Graph. It is also $2n$ vertices & $2n$ edges. The vertex set of G_2 are $\{1,2,3,\ldots.,n,N(1),N(2),N(3),\ldots\ldots.,N(n)\}$.

Now consider the Lexicographic Product of G_1 &G_2 is denoted by $G_1[G_2]=G$ (say), where the vertex is

$V(G)=V(G_1)\times V(G_2)=\{(a,1),(b,1),(c,1)\ldots,(m,1),(N(a),1),(N(b),1),\ldots,(N(m),1),$

$(a,2),(b,2),(c,2),\ldots.(m,2),(N(a),2),(N(b),2),\ldots,(N(m),2),\ldots,$

$(a,n),(b,n),(c,n),\ldots..,(m,n),(N(a),N(n)),(N(b),N(n)),\ldots\ldots(N(m),N(n))\}$.

$V(G)=\{(u_i,v_j);u_i\in V(G_1),v_j\in V(G_2)\}$, where $1\le i\le 2m$,$1\le j\le 2n$.

The total number of vertices in G is $2m\times 2n=4mn$.

In Lexicographic Product, any two vertices (u_i,v_j), $(u_{i'},v_{j'})$ are said to be adjacent in G. If (i) $u_i=u_{i'}$ and v_j is adjacent to $v_{j'}$ in G_2 (or) (ii) u_i is adjacent to $u_{i'}$ in G_1.

Case(i): from condition (i), for adjacency

$u_i = u_{i'}$ and v_j is adjacent to $v_{j'}$ in G_2.

Each vertex in G has $2n$ neighbours, which are adjacent in G_2.

Therefore, the degree of each vertex in G is k_2 . i.e., $d(u_i, v_j) = k_2$

Case (ii): For G_1, each vertex (u_i, v_j) has $2n$ adjacency vertices.

Each vertex $(u_i, v_j) \in G$ is adjacent to $k_1(2n)$ vertices $(u_{i'}, v_{j'})$, where $u_{i'}$ is adjacent to u_i in G_1.

Therefore each u_i, $2nk_1$ adjacent vertices $v_{j'}$ in G_1.

Then the degree of each vertex in G are $d(u_i, v_j) = 2nk_1$.

It observed that the above two cases (i) & (ii), we have

The degree of each vertex in G is $d(u_i, v_j) = 2nk_1 + k_2$.

Since, there are $4mn$ vertices in the Lexicographic Product Graph G.

So, the total degree of each vertex in G is

$$4mn \times \sum_{i=1}^{n} d(u_i, v_j) = 4mn(2nk_1 + k_2)$$

Let $\in$ be the total number of edges in G.

By handshaking theorem,

$$\sum_{i=1}^{n} d(u_i, v_j) = 2\in ◆ 2\in = 4mn(2nk_1 + k_2) ◆ \in = 2mn(2nk_1 + k_2).$$

EX: If $n = 3, m = 4$ (Figures 8 and 9)

MATLAB code

```
%MATLAB Code for Lexicographic Product of two Regular Open Neighborhood Graphs
%Input values for m, n, k1, and k2
m = input('Enter the value of m (m >= 3): ');
n = input('Enter the value of n (n >= 4): ');
k1 = input('Enter the value of k1 (k1 >= 2): ');
k2 = input('Enter the value of k2 (k2 >= 2): ');
%Validate inputs
if m < 3 || n < 4 || k1 < 2 || k2 < 2
error('Invalid input values. Ensure m >= 3, n >= 4, and k1, k2 >= 2.');
end
%Calculate total number of vertices in the Lexicographic Product Graph
total_vertices = 2 * m * 2 * n; % 4mn
%Calculate all edges in the Lexicographic Product Graph
total_edges = 2 * m * n * (2 * n * k1 + k2);
%Calculate degree of each vertex in the Lexicographic Product Graph
vertex_degree = 2 * n * k1 + k2;
%Display results
fprintf('\nResults for Lexicographic Product Graph G = G2[G1]:\n');
fprintf('m = %d, n = %d, k1 = %d, k2 = %d\n', m, n, k1, k2);
fprintf('Total vertices in G: %d\n', total_vertices);
```

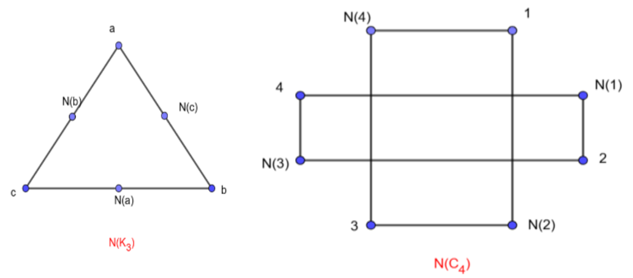

Figure 8: Open neighborhood graph $N(K_3)$ & open neighborhood graph $N(C_4)$

```
fprintf('Degree of each vertex in G: %d\n', vertex_degree);
fprintf('Total edges in G: %d\n', total_edges);
%Verify handshaking theorem
%Sum of degrees of all vertices in G
sum_degrees = total_vertices * vertex_degree;
%Total edges using handshaking theorem
edges_handshaking = sum_degrees / 2;
fprintf('Sum of degrees of all vertices: %d\n', sum_degrees);
```

Output

Results for Lexicographic Product Graph G = G2[G1]:
m = 3, n = 4, k1 = 2, k2 = 2
Total vertices in G: 48
Degree of each vertex in G: 18
Total edges in G: 432
Sum of degrees of all vertices: 864

Theorem

The total number of edges in lexicographic product graph $G = G_2[G_1]$ of two regular open neighborhood graphs G_1 & G_2 is $2mn(2nk_2 + k_1)$ where $m \geq 4, n \geq 3, k_1, k_2 \geq 2$.

Proof

Let G_1 be a k_1 Regular open neighborhood graph. Then it has $2m$ vertices, $2m$ edges and consider $\{1,2,3,\ldots,m, N(1), N(2), N(3),\ldots\ldots, N(m)\}$ be the vertices in G_1.

Let G_2 be another k_2 regular open neighborhood graph. It is also $2n$ vertices. Then vertex set of G_2 are $V(G_2) = \{a,b,c,\ldots\ldots,n, N(a), N(b), N(c),\ldots, N(n)\}$.

Now consider the lexicographic product of G_1 & G_2 is denoted by $G_2[G_1] = G$ (say), where the vertex is

$(2,a),(2,b),(2,c),\ldots(2,n),(2,N(a)),(2,N(b)),\ldots,(2,N(n)),\ldots,$

$V(G) = V(G_2) \times V(G_1) = \{(1,a),(1,b),(1,c)\ldots,(1.,n),(1,N(a)),(1,N(b)),\ldots,(1,N(n)),$

$(m,a),(m,b),(m,c),\ldots\ldots,(m,n),(N(m),N(a)),(N(m),N(b)),\ldots\ldots(N(m),N(n))\}$.

$V(G) = \{(u_i, v_j); u_i \in V(G_1), v_j \in V(G_2)\}$, where $1 \leq i \leq 2m$,$1 \leq j \leq 2n$.

The total number of vertices in G is $2m \times 2n = 4mn$.

In lexicographic product, any two vertices (u_i, v_j), $(u_{i'}, v_{j'})$ are said to be adjacent in G. If (i) $u_i = u_{i'}$ and v_j is adjacent to $v_{j'}$ in G_1 (or) (ii) u_i is adjacent to $u_{i'}$ in G_2.

Case(i): from condition (i), for adjacency of two vertices in G, $u_i = u_{i'}$ and v_j is adjacent to $v_{j'}$ in G_1. Each vertex in G has $2m$ neighbors, which are adjacent in G_1. Therefore, the degree of each vertex in G is k_1.

i.e., $d\left(u_i, v_j\right) = k_1$.

Case (ii): For G_2, each vertex $\left(u_i, v_j\right)$ has $2n$ adjacency vertices.

Each vertex $\left(u_i, v_j\right) \in G$ is adjacent to $(2n)k_2$ vertices $\left(u_{i'}, v_{j'}\right)$, where $u_{i'}$ is adjacent to u_i in G_2. Therefore each u_i, $(2n)k_2$ adjacent vertices $v_{j'}$ in G_2.

Then the degree of each vertex in G is $d\left(u_i, v_j\right) = 2nk_2$.

We conclude that in the above two cases (i) & (ii), we have

The degree of each vertex in G is $d\left(u_i, v_j\right) = 2nk_2 + k_1$

Since there are $4mn$ vertices in the Lexicographic Product Graph. So, the total degree of each vertex in G is

$$4mn \times \sum_{i=1}^{n} d\left(u_i, v_j\right) = 4mn\left(2nk_2 + k_1\right).$$

Let $\in$ be the total number of edges in G.

By handshaking theorem, $\sum_{i=1}^{n} d\left(u_i, v_j\right) = 2\in$�$2\in = 4mn\left(2nk_2 + k_1\right)$

$$\therefore \in = 2mn\left(2nk_2 + k_1\right).$$

MATLAB code

```
%MATLAB code for calculating properties of the lexicographic product graph G
%Input parameters
m = input('Enter the number of vertices in G1 (m >= 4): ');
n = input('Enter the number of vertices in G2 is (n >= 3): ');
k1 = input('Enter the degree of each vertex in G1 (k1 >= 2): ');
k2 = input('Enter the degree of each vertex in G2 (k2 >= 2): ');
%Validate input constraints
if m < 4 || n < 3 || k1 < 2 || k2 < 2
error('Input constraints not satisfied. Ensure m >= 4, n >= 3, k1 >= 2, and k2 >= 2.');
end
%Calculate properties of the lexicographic product graph G
Total Vertices = 2 * m * n;      %Total vertices in G
degG1 = k1;                      %Degree of each vertex in G1
degG2 = k2;                      %Degree of each vertex in G2
degG = 2 * n * k2 + k1;          %Degree of each vertex in G
total Edges = (total Vertices * degG) / 2;  %Total edges in G using handshaking lemma
%Display results
fprintf('Total Vertices in G : %d\n', total Vertices);
fprintf('Degree of each vertex in G1: %d\n', degG1);
fprintf('Degree of each vertex in G2 is : %d\n', degG2);
fprintf('Degree of each vertex in G : %d\n', degG);
fprintf('Total Edges in G : %d\n', total Edges);
```

OUTPUT

Enter the number of vertices in G1 (m >= 4): 4
Enter the number of vertices in G2 is (n >= 3): 3
Enter the degree of each vertex in G1 (k1 >= 2): 2
Enter the degree of each vertex in G2 (k2 >= 2): 2
Total Vertices in G : 24
Degree of each vertex in G1: 2
Degree of each vertex in G2 is: 2
Degree of each vertex in G : 14
Total Edges in G : 168

NOTE: The Lexicographic Product of two open neighborhood graphs are not commutative.

Conclusion

We have investigated in this paper the edge properties of the lexicographic product of graph with its open neighborhood graphs. The total number of edges in the lexicographic product of a graph and its open neighborhood graph has been determined using MATLAB implementation and validated in reverse. The lexicographic product of a graph with its open neighborhood graph is not commutative. i.e., $C_n\left[N(C_n)\right] \neq N(C_n)[C_n]$. The lexicographic product of two open neighborhood graphs is also not commutative.

Acknowledgments

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