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RESEARCH ARTICLE

Certain findings on the gamma graph of some graphs

Krupali Bhatt*, **Tushharkumar Bhatt**

Abstract

Suppose every vertex v ϵ V in graph G is either an element of subset S or is close to an element of subset S. In that case, subset S is said to be a dominant set of the vertex set V. If a set S is a γ-set. Its minimal cardinality is equal to the dominance number γ(G) of G. Any two vertices in a graph G are considered neighboring if their γ-sets vary by precisely one vertex. The gamma graph of G is denoted as γ · G, and its γ-sets serve as vertices in the graph. We discuss the gamma graph of path, cycle, ladder, and star graphs in the middle of the paper and look at a number of results pertaining to the gamma graph of pan and lollipop graphs.

Keywords: Dominating sets, Gamma sets, Gamma graph, Middle graph.

Introduction

For the duration of the article, we will be treating *G* as a basic finite undirected graph. The vertex set *V(G)* and the edge set *E(G)* make up a graph G. Our graph theory notation is based on the standard notation used in a book on the subject by Balakrishnan and Ranganathan (R. Balakrishnan and K. Ranganathan, 2012). Haynes et al. (T. W. Haynes, S. T. Hedetniemi and P. J. Slater, 2013) provide an overview of graph dominance theory. If every vertex v in V is either an element of S or is close to an element of S, then the set of vertices S⊆V in the graph *G=(V, E)* is termed a dominant set. A set S is called a γ-set, and its minimal cardinality is equal to the dominance number *γ(G)* of G. It was Sridharan and Subramanian who initially conceived of the Gamma graph. (N. Sridharan, S. Amutha, S. B. Rao, 2013) as a definition, the gamma graph of G is the graph with vertex set S, where S is the collection of all γ-sets in G and any two vertices S1 and S2 are near if |S_1∩S_2 |=*γ(G)-1*. This graph is denoted as γ · G. Gamma graphs have been the subject of much investigation. Bien examined the gamma graphs of specific types of trees. (A. Bien) Anusuya and Kala (V.Anusuya and R. Kala, 2015) examined modified γ graphs, *G(γ_m)* of certain grid graphs, and Lakshmanan et al. (S. A. Lakshmanan, A. Vijayakumar, 2010) addressed the fact that the cartesian product closes the collection of all gamma graphs. For every given graph, Fricke et al. (G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, K. R. Huston, 2011) have discovered that every tree is actually a gamma graph." The inductive approach to finding the gamma graph of cycle *C_(3k+1)* was brought up by Isaac and Bhatt. (R. Isaac and K. Bhatt, 2023) The expression *γ·C_(3k+1)* is 4-regular, as they found.

The concept of the middle graph M(G) of a graph was introduced by Hamada and Yoshimura (T. a. I.Yoshimura, 1976) as an intersection graph on the vertex set of G. In this paper, we investigate results on the gamma graph of the middle graph of a path, cycle, ladder graph and star graph. We aim to expand the understanding of gamma graphs by presenting new results and properties.

Department of Mathematics, Atmiya University, Rajkot, Gujarat, India.

***Corresponding Author:** Krupali Bhatt, Department of Mathematics, Atmiya University, Rajkot, Gujarat, India, E-Mail: krupalibhatt1612@gmail.com

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Main Results

Definition

The middle graph of a graph G = (V,E) denoted by M(G), is a graph in which the set of vertices is V∪E and two vertices x, y in M(G) are adjacent if and only if one of the following holds.

- $x, y \in E$ and x, y are adjacent in G.
- $x \in V, y \in E$ and x, y are incident in G.

Proposition

(S. Finbow, C.M. van Bommel, 2019) For any path P_n of order $n \geqslant 2$, we have $\gamma(M(P_n)) = \lceil \frac{n}{2} \rceil$.

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Theorem

$\gamma \cdot M(P_n)$ where $n = 2k+1; k \geq 2$ is isomorphic to the graph G of order n in which 2 vertices are of degree 1 and 3, $\left\lceil \frac{n}{2} \right\rceil - 1$ vertices of degree 2 and remaining $\left\lceil \frac{n}{2} \right\rceil - 2$ vertices of degree 4.

Proof. Let $V = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots e_{n-1}\}$ Be the vertex set of $M(P_n)$. By proposition 2.1, Consider $S_1 = \{e_1, e_2, e_4, e_6, \ldots, e_{n-1}\}$ as one of the γ-set with cardinality k+1. Also, e_1 and e_2 are adjacent.

For finding other -sets we cannot select e_2 & e_3, e_4 & e_5 as an djacent pair of vertices. So the γ-sets are, $S_2 = \{e_1, e_3, e_4, \ldots, e_{n-1}\}, S_3 = \{e_1, e_3, e_5, e_6, \ldots, e_{n-1}\}, \ldots, S_n = \{e_1, e_3, e_5, \ldots, e_{n-1}\}$. We get, $\frac{n-1}{2}$ such γ-sets. That means the number of γ-sets is k To form other γ-sets, each e_i & e_{i+1} of $S'_i s; 1 \leq i \leq n$ can be replaced one by one with its adjacent $v'_i s$ say, $S_{n+1} = \{v_1, e_2, e_4, \ldots, e_{n-1}\}, S_{n+2} = \{e_1, v_3, e_4, \ldots, e_{n-1}\}, \ldots$ proceeding in this way, some of the γ-sets are repeating. The number of such γ-sets is k+1.

Therefore, the total number of γ-sets is 2k+1.

By the definition of a gamma graph, the γ-sets are adjacent if and only if they differ by one vertex. So, γ·M(P_n)=G.

Illustration

Given figure 1 shows the gamma graph of a middle graph of P_{11}.

Corollary

$\gamma \cdot M(P_n) = K_1$, where $n = 2k; k \geqslant 1$.

Proof. Assume that the vertex set $V = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots e_{n-1}\}$ of $M(P_n)$.

Consider $S = \{e_1, e_3, e_5, e_7, \ldots, e_{n-1}\}$ as the γ-set with cardinality k and we have a unique gamma set.So,

$$\text{So}, \gamma \cdot M(P_n) = K_1.$$

Theorem $\gamma \cdot M(C_n) = \begin{cases} \overline{K_2}, if\ n = 2k \\ M(C_n),\ if\ n = 2k+1 \end{cases}; for\ k \geq 1.$

Proof. Let $V = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots e_n\}$ be the vertex set of $M(C_n)$.

Case (i): *For* $n = 2k$, $\gamma(M(C_{2k})) = \left\lceil \frac{2k}{2} \right\rceil = k$.

Consider $S_1 = \{e_1, e_3, e_5, \ldots, e_{n-1}\}$ as γ-set with cardinality k. For finding other γ-sets we cannot replace any of the $e'_i s$ with its adjacent vertex. (∵ $\gamma(M(C_{2k})) > k$ which is not possible).

So, only two γ-sets are possible S_1 and $S_2 = \{e_2, e_4, e_6, \ldots, e_n\}$ also they are disjoint γ-sets.

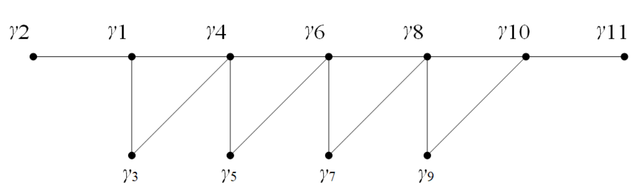

Figure 1: Gamma graph of $M(P_{11})$

Therefore, by definition of the gamma graph S_1 and S_2 are non-adjacent.

Hence, $\gamma \cdot M(C_n) = \overline{K_2}$.

Case (ii): *For* $n = 2k+1$, $\gamma(M(C_{2k+1})) = \left\lceil \frac{2k+1}{2} \right\rceil = k+1$.

Consider $S = \{e_i, e_{i+1}, e_{i+3}, \ldots, e_{i+(n-2)}\}$ as γ-set of cardinalities k+1 with one of the adjacent pair of $e'_i s$.

The cycle is of length n. So, n such γ-sets can be form. Say, $S_1 = \{e_1, e_2, e_4, \ldots, e_{n-1}\}, S_2 = \{e_2, e_3, e_5, \ldots, e_n\}, S_3 = \{e_1, e_3, e_4, e_6, \ldots, e_{n-1}\}, \ldots, S_{n-1} = \{e_2, e_4, e_6, \ldots, e_{n-1}, e_n\}, S_n = \{e_1, e_3, e_5, \ldots, e_{n-2}, e_n\}$.

For finding other γ-sets, each $e_i\ and\ e_{i+1}$ of $S'_i s; 1 \leq i \leq n$ can be replaced one by one with its adjacent $v'_i s$ say, $S_{n+1} = \{v_1, e_2, e_4, \ldots, e_{n-1}\}, S_{n+2} = \{v_2, e_3, e_5, \ldots, e_n\}, \ldots S_{2n} = \{e_1, e_3, e_5, \ldots, e_n\}$, proceeding in this way, some of the γ-sets are repeating.

So, the total number of γ-sets is 2*n*.

Based on the definition of a Gamma graph, each $S'_i s; 1 \leq i \leq n$ are adjacent to other two $S'_i s$ and two $S'_j s; n+1 \leq j \leq 2n$.

Also, each $S'_j s$ are adjacent to two $S'_j s$.

Hence, $\gamma \cdot M(C_{2k+1}) = M(C_{2k+1})$.

Definition

(Christopher M. Bommel, 2019) The ladder graph L_n is defined as $L_n \square K_2$ where P_n is a path with n vertices cartesian product with K_2 a complete graph with two vertices.

Theorem $\gamma \cdot M(L_n) = \overline{K_n}; n > 1$.

Proof. Let *V* be the vertex set of $M(L_n)$. $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{v_1, v_2, \ldots, v_n\}, V_2 = \{u_1, u_2, \ldots, u_n\}$ and $V_3 = \{e_{11}, e_{12}, e_{22}, e_{21} \ldots, e_{nn}\}$.

Consider $S_1 = \{e_{11}, e_{22}, e_{33}, \ldots, e_{nn}\}$ as one of the γ-set with cardinality n. For finding other γ-sets we select vertices from V_3.

These γ-sets are the vertices of the gamma graph which are non-adjacent. So, it is the null graph of order n.

Theorem $\gamma \cdot M(K_{1,n}) = K_1; n > 1$.

Proof. Let $V = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots e_n, w\}$ be the vertex set of $M(K_{1,n})$.

Consider $S = \{e_1, e_2, e_3, \ldots, e_n\}$ as the γ-set with cardinality n and we have a unique gamma set.

So, $\gamma \cdot M(K_{1,n}) = K_1$

Definition

(C. M. Mynhardt and L. Trshima, 2017) *n*-**pan graph** is the graph obtained by joining a cyclic graph C_n to a singleton graph K_1 with a bridge.

Proposition

(E. Connelly, S. T. Hedetniemi and K. R. Huston, 2011) =

$$\gamma(n - pan\ graph) = \begin{cases} \frac{n}{3};\ if\ n = 3k \\ \left\lceil \frac{n}{3} \right\rceil;\ otherwise \end{cases}$$

Theorem

$$\gamma \cdot n - pan\ graph = \begin{cases} K_1,\ \ if\ n = 3k \\ (n-1) - pan\ graph, if\ n = 3k+1 \\ P_{k+1}, if\ n = 3k+2 \end{cases}; \text{for } k \geq 1.$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ be the vertex set of n-pan graph.

Case (i): For $n = 3k$,

Consider $S = \{v_2, v_5, \ldots, v_{n-1}\}$ as γ-set of cardinality same as domination number. As we have a unique γ-set, by definition of gamma graph $\gamma \cdot n - pan\ graph = K_1$.

Case (ii): For $n = 3k + 1$,

Let $S_1 = \{v_1, v_4, \ldots, v_n\}$ and $S_2 = \{v_2, v_3, \ldots, v_{n-1}\}$ be two γ-sets in which S_2 has one pair of adjacent vertices. For finding other γ-sets we fix $(\lceil \frac{n}{3} \rceil - 1)$ vertices of γ-sets. With the vertices from S_1, we can't find other γ-sets as v_1 is a pendant vertex so the other γ-sets are from S_2. Also, the γ-sets are vertices of the gamma graph and by the definition of the gamma graph other than S_1 all are of degree 2. Therefore, the gamma graph of $n - pan\ graph$ is $(n-1) - pan\ graph$.

Case (iii): For $n = 3k + 2$,

Consider $S_1 = \{v_2, v_5, \ldots, v_{n-1}\}$ as one of the γ-sets. For finding other γ-sets we fix $(\lceil \frac{n}{3} \rceil - 1)$ vertices of S_1 and we get k + 1, γ-sets.

So, by definition, the gamma graph of n-pan graph is a path of length k + 1.

Definition

(E. Connelly, S. T. Hedetniemi and K. R. Huston, 2011) The lollipop graph is a graph obtained by joining a complete graph K_n to a path P_m with a bridge. It is denoted by $L_{m,n}$.

Theorem 2.6. $\gamma \cdot L_{m,n} = \begin{cases} L_{\frac{m}{3},n}, if\ n = 3k \\ K_1,\ if\ n = 3k+1 \end{cases}; for\ k \geq 1.$

Proof. Let $V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$ be the vertex set of $L_{m,n}$.

Case (i): For $= 3k$, $\gamma(L_{m,n}) = \lceil \frac{m+2}{3} \rceil$.

Let $S_1 = \{v_1, v_3, \ldots, v_{n-2}, u_1\}$ and $S_2 = \{v_1, v_4, \ldots, v_{n-1}, u_1\}$ be two γ-sets in which S_1 has one pair of adjacent vertices. For finding other γ-sets we fix $(\lceil \frac{m+2}{3} \rceil - 1)$ vertices of γ-sets. With the vertices from S_1 we can get only one γ-set as v_2 replace the vertex v_1 and the other γ-sets are from S_2. Also, the γ-sets are vertices of the gamma graph and by the definition of the gamma graph other than S_1 all are of degree n.

Therefore, the gamma graph of $L_{m,n}$ is a lollipop graph with path of length $\frac{m}{3}$ and complete graph of order n.

Case (ii): For $n = 3k + 1$,

Consider $S = \{v_2, v_5, \ldots, v_{n-2}, u_1\}$ as the unique γ-set, so, $\gamma \cdot L_{m,n} = K_1$.

Conclusion

We have initiated a study on the gamma graph of the middle graph of paths, finding that the resultant graph retains the same order. Additionally, our research reveals that the gamma graph of the middle graph of an odd cycle is isomorphic to itself,while that of an even cycle, ladder graph and star graph are isomorphic to a null graph with two vertices, complement of a complete graph of same order and a trivial graph respectively. Also, we have identified the gamma graph of the pan graph and the lollipop graph.

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