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RESEARCH ARTICLE

Extended fractional derivative: Some results involving classical properties and applications

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Abstract

In this paper, considering the classical definition of derivative in the limit form, we can define a new fractional derivative called the extended fractional derivative, which depends upon two parameters α and a. For this new fractional derivative, we have proved the properties of classical derivatives such as the product rule, the quotient rule, the chain rule, Rolle's and Lagrange's mean value theorems, etc., which are nor satisfied by the existing fractional derivatives defined by Riemann Liouville, Caputo, Grunwald. Moreover, we have defined the extended fractional integral and proved some related results. Solve extended fractional differential equation.

Keywords: Caputo fractional derivative, Riemann-Liouville fractional derivative, Extended fractional derivative, Conformable fractional derivative.

MSC2010-Mathematics Subject Classification: 34B05, 34A12,

Introduction

The theory of fractional calculus has attracted the attention of many researchers due to the broad and successful applicability of the theory of differential equations in the fields of applied mathematics, mathematical physics, chemical sciences, biological sciences, engineering and technology, etc. This is because of the applicability of the derivatives and integrals of the fractional order with the corresponding initial and boundary conditions. The theory of fractional calculus is applied in the fields of fluid dynamics, electromagnetism, viscoelasticity, feedback amplifier and capacitor analysis, etc., in addition to the technologies and fields already mentioned. In recent decades, a number of academics have pointed out how important fractional order differentials and integrals are in understanding the viscoelastic properties of real materials, such as polymers. The derivative of non-integer order has been an intriguing field of research for many centuries. Among the fractional derivative types introduced were Riemann-Liouville, Caputo, Hadamard, Gr¨unwald-Letnikov, Marchaud, and Riesz as given in Miller, K. S. and Ross, B (1993), Oldham K., Spanier J, (1974), Podlubny, I. 1999. Most fractional derivatives are defined using fractional integrals. The characteristics of the traditional integer order derivatives are not satisfied by any of these fractional derivatives. We know that a constant's integer order derivative is zero, and we anticipate that the same will hold true for fractional derivatives as well. However, with the exception of the Caputo fractional derivative, this is not the case for the majority of fractional derivatives. Additionally, fractional derivatives do not satisfy the properties of classical derivatives, such as the Product rule, the Quotient rule, the Chain rule, Rolle's and Lagrange's mean value theorems. Numerous authors have made significant contributions in this direction. They have defined a few new fractional derivatives and proven some of the properties that the earlier fractional derivatives did not satisfy (Katugampola U. N. 2014, Khalil R.2014, Abdljawad *et al*. 2015, Hammad M. *et al*. 2014, Almeida, R. *et al* 2017).

These conformable derivatives are all some extensions of the definition of the classical limit form.

A new conformable fractional derivative and applications is given by Ahmed Kajouni, Ahmed Chafiki, Khalid Hilal, and Mohamed OukessouA in 2021.

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The exponential definition of fractional derivative is given by Kamble *et al*. (2024).

A boundary value issue involving Caputo non-linear fractional integrodifferential equations of order $0 < \beta \le 1$ and $0 < \alpha \le 1$ has been studied by the authors Kamble *et al*.(2023).

In order to accomplish this, the writers have used the fixed point theory to support a number of uniqueness and existence claims. In particular, they have used the Banach contraction mapping principle and, in some weak cases, Krasnoselskii's fixed point theorem. They provide several examples to back up their findings.

Basic of q-fractional derivatives is given by Acka, H., Benbourenane, J., Eleuch, H. and Sanh, Z (2019).

Abels's formula and Wronskian for conformable fractional differential equations is given by Hammad M. and Khalil R.(2014).

A remark on local fractional calculus and ordinary derivatives is given by almeida, m. guzowska, and t. odzijewicz in 2017.

Gohar Fractional Derivative

Theory and applications is given by Gohar A., Younes M., Doma S.B. (2023).

The basic theory of first-order derivative is given in Robert G. bartle, Donald R. sherbert (2000).

Methods and Materials

R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh 2014 give the definition known as of comfortable fractional derivative and Conformable fractional integral as follows:

Definition 1.1: Comformable Fractional Derivative

Let $f:[0,\infty)\to R$, then the comfortable fractional derivative of f at $t>0$ is

$$D^{\alpha}(f)(t)=\lim_{h\to 0}\frac{f\left(t+ht^{(1-\alpha)}\right)-f(t)}{h},\alpha\in(0,1]$$

if the limit exist then f is called α comformable differentiable at t ,

if f is α- comformable differentiable in some (0, b), b >0 and $\lim_{t\to 0^+} f^{\alpha}(t)$,exists then define $\lim_{t\to 0^+} f^{\alpha}(t)=f^{\alpha}(0)$.

and extended the definition for $n<\alpha\le n+1, n\in N$

$$D^{\alpha}(f)(t)=\lim_{h\to 0}\frac{f^{\alpha-1}\left(t+ht^{(\alpha-\alpha)}\right)-f^{\alpha-1}(\tau)}{h}$$

Where α is the smallest integer greater than or equal to .=n+1).

$$D^{\alpha}(f)(t)=\lim_{h\to 0}\frac{f^{n}\left(t+ht^{(n+1-\alpha)}\right)-f^{n}(t)}{h}$$

provided the limit exist.

Definition 1.2: Conformable fractional integral:

let α ∈ (0, 1] and t > 0 let f be a function defined on [0, ∞) then the α comfortable fractional integral is given by

$$I^{\alpha}f(t)=\int_0^t\frac{f(s)}{s^{1-\alpha}}ds.$$

Provided the integral exists.

Conformable fractional derivative was generalized by U. N. Katugampola in 2014 and gives a new definition of a fractional derivative with the same conformable fractional integral as follows

Definition 1.3 New Definition of Fractional Derivative

$f:[0,\infty)\to R$, then the new fractional derivative of f at t $h>0$ is

$$D^{\alpha}(f)(t)=\lim_{h\to 0}\frac{f\left(te^{\mu t^{-\alpha}}\right)-f(t)}{h},\alpha\in(0,1)$$

Provided the limit exist.

And proved the classical properties with same integral.

The purpose of this work is to further generalize the results obtained by U. N. Katugampola in 2014 and introduce a new fractional derivative of two parameters as the most natural extension of the familiar limit definition of the derivative of a function f at a point.

And given name extended fractional derivative and inverse operation, i.e., extended fractional integral, is defined.

Observations/Main Results

Extended Fractional Derivative

In this section, we have given another natural extension of the limit form derivative and we call it the extended fractional derivative and proved the properties of classical derivatives such as the product rule, the quotient rule, the chain rule, Rolle's and Lagrange's mean value theorems, etc. which are nor satisfied by the existing fractional derivatives defined by Riemann-Liouville, Caputo, Grunwald-Letnikov, Marchaud, and Riesz. First, we define the extended fractional derivative as follows.

Definition

(first form)Let f [0, ∞) -→ R be any real-valued function and $\alpha\in(0,1], a>0, a\in R, h>0$. We define (α,a) Extended fractional derivative of f of order α at $t\in[0,\infty)$, denoted $D_a{}^{\alpha}f(t)=(f_a)^{\alpha}(t)=T^{\alpha}{}_a f(t)$, by

$$D_a{}^{\alpha}f(t)=(f_a)^{\alpha}(t)=T^{\alpha}{}_a f(t)=\lim_{h\to 0}\frac{f\left(ta^{ht^{-\alpha}}\right)-f(t)}{h}$$

for all $t>0$, Provided the limit exists.

The maclaurin series expansion of $a^{ht^{-\alpha}}$ is

$$a^{ht^{-\alpha}} = 1 + ht^{-\alpha}\ln a + O(h)^2$$

(Second form):

$$D_a^{\alpha}f(t) = \left(f_a\right)^{\alpha}(t) = T_a^{\alpha}f(t) = \lim_{h\to 0}\frac{f\left(t\left(1+ht^{-\alpha}\ln a + O(h)^2\right)\right) - f(t)}{h}$$

for all $t > 0$, Provided the limit exist.

Remark 3.1: In case the limit exists, we say that f is (α, a)-extended fractionally differentiable (EFD).

If f is (α, a) - extended fractional differentiable in some (0, c), c > 0, and $\lim_{t\to 0^+} f_a^{\alpha}(t)$ exists, then we define . $\lim_{t\to 0^+} f_a^{\alpha}(t) = f_a^{\alpha}(0)$

Remark 3.2: It can be observed that the definition 2.1 turns out to be the classical definition of the derivative of first order for the special case α = 1 and a = e.

Remark 3.3: Setting a = e in the definition 2.1, we get the definition of the new fractional derivative as defined by U. N. Katugampola in 2014.

Remark: 3.4: The extended Fractional derivative is depends on both the order 'α' and the point 'a'.

Remark: 3.5: The same order derivative is different for different values of 'a'.

- *Theorem 3.1:*

The extended fractional derivative $D_a^{\alpha}f$ is a generalization of the q-derivative.

Proof: By definition of the extended fractional derivative(Second form), we have

$$D_a^{\alpha}f(t) = \lim_{h\to 0}\frac{f\left(t(1+h(\ln a)t^{(-\alpha)}\right) + o(h^2) - f(t)}{h},$$

$$\alpha\in(0,1], a>0, h>0, a\in R$$

Putting $q = 1 + h(\ln a)t^{(-\alpha)} + o(h^2), q\to 1 \text{ as } h\to 0$

$$D_a^{\alpha}f(t) = (\ln a)\lim_{h\to 0}\frac{f(tq)-f(t)}{qt^{\alpha}-t^{\alpha}},\ \alpha\in(0,1], a>0, a\in R$$

Remark 3.6., i.e., The extended fractional derivative $D_a^{\alpha}f(t)$ is a generalization of the q derivative.

Remark 3.7:Setting a = e we get

$$D_e^{\alpha}f(t) = \lim_{h\to 0}\frac{f(tq)-f(t)}{qt^{\alpha}-t^{\alpha}},\ \alpha\in(0,1], a>0, a\in R$$

Remark 3.8: Setting a = 2 we get

$$D_2^{\alpha}f(t) = (\ln 2)\lim_{h\to 0}\frac{f(tq)-f(t)}{qt^{\alpha}-t^{\alpha}},\ \alpha\in(0,1], a>0, a\in R$$

Remark 3.9:Setting a = e , we get .

$$D_e^{1}f(t) = \frac{df}{dt} = \lim_{h\to 0}\frac{f(tq)-f(t)}{qt-t},\ \alpha\in(0,1], a>0, a\in R$$

Now we prove a few elementary results. In the following theorems, we will assume that $f : [0,\infty) \to R$ is a real valued function, α ∈ (0, 1], a ∈ R,a>0 unless otherwise stated.

- *Theorem 3.2: (Relation Between Classical Derivative and EFD)*

$$D_a^{\alpha}f(t) = (\ln a)t^{1-\alpha}\frac{df(t)}{dt}$$

Proof: By definition of the extended fractional derivative, we have

$$\lim_{h\to 0}\frac{f\left(t + h(\ln a)t^{(1-\alpha)} + o(h^2)\right) - f(t)}{h}$$

Let $\pi = h(\ln a)t^{(1-\alpha)} + o(h^2)$ implies $h = \frac{\pi}{(\ln a)t^{(1-\alpha)}}$ as $h\to 0, \pi\to 0$

$$= \lim_{\pi\to 0}\frac{f(t+\pi)-f(t)}{\frac{\pi}{(\ln a)t^{(1-\alpha)}}}$$

$$(\ln a)t^{(1-\alpha)}\lim_{\pi\to 0}\frac{f(t+\pi)-f(t)}{\pi}$$

$$= (\ln a)t^{(1-\alpha)}\frac{df(t)}{dt}$$

Remark 3.10:

Setting a = e we get

$$D_e^{\alpha}f(t) = t^{1-\alpha}\frac{df(t)}{dt}$$

Remark 2.11:

Setting a = 2 we get

$$D_e^{\alpha}f(t) = (\ln 2)t^{1-\alpha}\frac{df(t)}{dt}$$

- *Theorem3. 3: (α, a) Extended Fractional Derivative of Some Elementary Functions.*

Let $\alpha\in(0,1), a>0, a\in R, \lambda\in R$ then

1. $D_a^{\alpha}\left(t^{\lambda}\right) = \lambda t^{\lambda-\alpha}(\ln a)\quad \forall\lambda\in R$

2. $D_a^{\alpha}(\lambda)=0$

3. $D_a^{\alpha}(\lambda)=0,\forall\lambda\in R$

4. $D_a^{\alpha}\left(e^{\lambda t}\right)=\lambda t^{\lambda-\alpha}(\ln a)e^{\lambda t},\lambda\in R$

5. $D_a^{\alpha}\left(a^{\lambda t}\right)=\lambda t^{\lambda-\alpha}(\ln a)^2 a^{\lambda t},\lambda\in R$

6. $D_a^{\alpha}(\sin\lambda t)=\lambda t^{\lambda-\alpha}(\ln a)\cos\lambda t,\lambda\in R$

7. $D_a^{\alpha}(\cos\lambda t)=-\lambda t^{\lambda-\alpha}(\ln a)\sin\lambda t,\lambda\in R$

8. $D_a^{\alpha}(\tan(\lambda t))=\lambda t^{\lambda-\alpha}(\ln a)\sec^2(\lambda t),\lambda\in R$

9. $D_a^{\alpha}(\cot(\lambda t))=-\lambda t^{\lambda-\alpha}(\ln a)\mathrm{cosec}^2(\lambda t),\lambda\in R$

10. $D_a^{\alpha}(\cot(\lambda t))=-\lambda t^{\lambda-\alpha}(\ln a)\mathrm{cosec}^2(\lambda t),\lambda\in R$

11. $D_a^{\alpha}(\mathrm{Cosec}(\lambda t))=-\lambda t^{\lambda-\alpha}(\ln a)\mathrm{Cosec}(\lambda t)\cot(\lambda t),\lambda\in R$

The proofs follows from the theorem (3.2) and classical derivatives of the elementary functions given in Robert G. bartle, Donald R. sherbert (2000), introduction to real analysis .

- *Theorem3. 4. (Differentiability Implies Continuity)*

If a function : [0, ∞) -→ R is (α,a) Extended fractional differentiable at $t_0>0$

$\alpha\in(0,1]$, then is continuous at $t_0\in[0,\infty)$.

Proof. Consider the following equation

$$f\left(t_0+h(\ln a)t_0^{(1-\alpha)}+o(h^2\right)-f(t_0)$$
$$=\frac{f\left(t_0+h(\ln a)t_0^{(1-\alpha)}+o(h^2)\right)-f(t_0)}{h}\times h$$

then

$$\lim_{h\to0}f\left(t_0+h(\ln a)t_0^{(1-\alpha)}\right)-f(t_0)$$
$$=\lim_{h\to0}\frac{f\left(t_0+h(\ln a)t_0^{(1-\alpha)}\right)-f(t_0)}{h}.\lim_{h\to0}h$$

Since f is (α, a)-extended fractional differentiable function at a point $t_0,D_a^{\alpha}f(t)$ exists. Hence we have

$$\lim_{h\to0}f\left(t_0+h(\ln a)t_0^{(1-\alpha)}\right)-f(t_0)=,D_a^{\alpha}f(t_0).0$$
$$\lim_{h\to0}f\left(t_0+h(\ln a)t_0^{(1-\alpha)}\right)=f(t_0)$$

Hence f is continuous t_0 .

- *Theorem.3.5.(Classical Properties)*

Let α ∈ (0, 1] and f,g be (α,a) Extended fractional differentiable at a point $t>0$. Then

(i) $D_a^{\alpha}(cf\pm dg)=cD_a^{\alpha}(f)\pm dD_a^{\alpha}(g)$, for all , $c,d\in R$ and f,g in the domain of . D_a^{α}

(ii) $D_a^{\alpha}(fg)=f\,D_a^{\alpha}(g)+gD_a^{\alpha}(f)$

(iii) $D_a^{\alpha}\left(\frac{f}{g}\right)=\frac{gD_a^{\alpha}(f)-fD_a^{\alpha}(g)}{g^2}$

proofs: The proofs follows from the theorem (3.2) and classical derivatives of the elementary functions given in Robert G. bartle, Donald R. sherbert (2000), introduction to real analysis.

- *Theorem:3.6 (Chain Rule for (α, α) Extended fractional derivative)*

Suppose I And J be open intervals in R^+, for $t\in[0,\infty)$, $t>0$, let $f,\,g:[0,\infty)\to R$ be two functions and f is (α, α) Extended fractional derivative at t and g is differentiable at $f(t)$ .then (gof) is (α,a) Extended fracrtion derivative at t and. $D_a^{\alpha}(gof)(t)=g'(f(t))D_a^{\alpha}f(t)$

(where (gof) isthe comosition of g and g′ is the first order derivative of g)

Proof: The proof follows from the theorem 3.2 and the chain rule for classical derivative given in Robert G. bartle, Donald R. sherbert (2000), introduction to real analysis.

- *Theorem 3.7: useful formulae for (α,a) extended fractional derivative: for $\alpha\in(0,1),a>0,a\in R$*

1. $D_a^{\alpha}\left(\frac{t^{\alpha}}{\alpha(\ln a)}\right)=1.$

2. $D_a^{\alpha}\left(e^{\left(\frac{t^{\alpha}}{\alpha(\ln a)}\right)}\right)=e^{\left(\frac{t^{\alpha}}{\alpha(\ln a)}\right)}.$

3. $D_a^{\alpha}\left(\sin\frac{t^{\alpha}}{\alpha(\ln a)}\right)=\cos\frac{t^{\alpha}}{\alpha(\ln a)}$.

4. $D_a^{\alpha}\left(\cos\frac{t^{\alpha}}{\alpha(\ln a)}\right)=-\sin\frac{t^{\alpha}}{\alpha(\ln a)}.$

Proof 1)

$$D_a^{\alpha}\left(\frac{t^{\alpha}}{\alpha(\ln a)}\right)=\frac{1}{\alpha(\ln a)}\lim_{h\to 0}\frac{(t+h(\ln a)t^{(1-\alpha)}+o(h^2))^{\alpha}-t^{\alpha}}{h}$$

$$=\frac{1}{\alpha(\ln a)}\lim_{h\to 0}\frac{t^{\alpha}+\alpha t^{\alpha-1}(\ln a)t^{(1-\alpha)}h+\frac{\alpha(\alpha-1)}{2}t^{\alpha-2}h^2(\ln a)^2\left(t^{(1-\alpha)}\right)^2+\ldots+-t^{\alpha}}{h}$$

$$=\frac{1}{\alpha(\ln a)}\lim_{h\to 0}\frac{\alpha t^{\alpha-1}(\ln a)t^{(1-\alpha)}h+\frac{\alpha(\alpha-1)}{2}t^{\lambda-2}h^2(\ln a)^2\left(t^{(1-\alpha)}\right)^2+\ldots}{h}$$

$$=\frac{\alpha(\ln a)}{\alpha(\ln a)}=1.$$

- *Defination 3.2.:*

Let f [0, ∞) -→ R be any real valued function $\alpha \in(n,n+1]$, $n\in N$, $a>0, a\in R, t\in[0,\infty)$ $h>0$. We define (α,a) Extended fractional derivative of f of order α at $t\in[0,\infty)$,denoted $D_a^{\alpha}f(t)=(f_a)^{\alpha}(t)=T^{\alpha}_a f(t)$, by for $n<\alpha\le n+1$

$$D_a^{\alpha}(t)=\lim_{h\to 0}\frac{D^{\alpha-1}\left(t+h(\ln a)t^{(\alpha-\alpha)}\right)-D^{\alpha-1}(t)}{h}$$

Where α is the smallest integer greater than or equal to α.

$$(\alpha=n+1)$$

or

$$D_a^{\alpha}(f)(t)=\lim_{h\to 0}\frac{D^{n}\left(t+h(\ln a)t^{(n+1-\alpha)}\right)-D^{n}(t)}{h},$$

Where D^n is the n-th order ordinary derivative at t given in Robert G. bartle, Donald R.sherbert (2000), introduction to real analysis.

Remark 3.10: If f is (α,a)- extended fractional differentiable in some (0, c), c > 0, and $\lim_{t\to 0^+} f_a^{\alpha}(t)$ exists then we define $\lim_{t\to 0^+} f_a^{\alpha}(t)=f_a^{\alpha}(0)$.

Remark:3.11 follows directly from definition

$$T^{\alpha}_a(f)=(\ln a)t^{(\alpha-\alpha)}\frac{d^{\alpha}f}{(dt)^{\alpha}}$$

Remark 3.13: for $(\alpha,2)$ extended fractional derivative we have

$$T^{\alpha}_2(f)=(\ln 2)t^{(\alpha-\alpha)}\frac{d^{\alpha}f}{(dt)^{\alpha}}$$

Remark 3.14: for (α,e) extended fractional derivative we have

$$T^{\alpha}_2(f)=(\ln 2)t^{(\alpha-\alpha)}\frac{d^{\alpha}f}{(dt)^{\alpha}}$$

Theorem3.8: Rolle's theorem for (α,a) Extended fractional differentiable functions: Let $a,c>0$ and $f:[c,d]\to R$

be a given function that holds the following

1. f is continuous in $[c,d]$,
2. f is (α,a) extended fractional differentiable for some $\alpha\in(0,1)$ on (c,d)

$$f(c)=f(d)$$

Then there exist $\varepsilon\in(c,d)$ such that $f_a^{\alpha}(\varepsilon)=0$

Proof: Since f is continuous in $[c,d]$ $f(c)=f(d)$, there is $\varepsilon\in(c,d)$, Which is point of local exrema.with no loss of generality.assume α is a point of local minimum. So

$$f^{\alpha}(\varepsilon)=\lim_{h\to\varepsilon^+}\frac{f\left(\varepsilon+h(\ln a)\varepsilon^{(1-\alpha)}+o(h^2)\right)-f(\varepsilon)}{h}$$

$$=\lim_{h\to\varepsilon^-}\frac{f\left(\varepsilon+h(\ln a)\varepsilon^{(1-\alpha)}+o(h^2)\right)-f(\varepsilon)}{h}$$

One limit is positive and other is negative so

$$f_a^{\alpha}(\varepsilon)=0$$

Theorem3.9:Mean value theorem for (α,a) Extended fractional differentiable functions :Let $a,c>0$ and $f:[c,d]\to R$

be a given function that holds the following

1. f is continuous in $[c,d]$,
2. f is (α,a) extended fractional differentiable for some $\alpha\in(0,1)$ on (c,d)

Then there exist $\varepsilon\in(a,b)$ such that

$$f_a^{\alpha}(\varepsilon)=\frac{f(d)-f(c)}{\frac{\delta^{\alpha}}{\alpha(\ln a)}-\frac{\gamma^{\alpha}}{\alpha(\ln a)}}$$

Proof: Consider the function

$$g(t)=f(t)-f(c)-\frac{f(d)-f(c)}{\frac{\delta^{\alpha}}{\alpha(\ln a)}-\frac{\gamma^{\alpha}}{\alpha(\ln a)}}\left[\frac{\tau^{\alpha}}{\alpha(\ln a)}-\frac{\gamma^{\alpha}}{\alpha(\ln a)}\right]$$

Then $g(c)=g(d)$, therefore the function g satisfies the condition of Rolles theorem

And using $f_a^{\alpha}\left(\frac{t^{\alpha}}{\alpha(\ln a)}\right)=1$ we get the result.

- *Theorem3.10:*
  Extended Mean value theorem for (α,a) Extended fractional differentiable functions :Let $a,c>0$ and $f,g:[c,d]\to R$

  be a given function that holds the following
  1. f,g is continuous in $[c,d]$,
  2. f,g is (α,a) extended fractional differentiable for some $\alpha\in(0,1)$

  Then there exist $\varepsilon\in(c,d)$ such that $\frac{f_a^{\alpha}(\varepsilon)}{g_a^{\alpha}(\varepsilon)}=\frac{f(d)-f(c)}{g(d)-g(c)}$

  Proof: Proof follows directly from theorem 2.9
- *Theorem 3.11 :*
  Let $a,c>0$ and $f,:[c,d]\to R$

  be a given function that holds the following

  f is continuous in $[c,d]$,

  f is (α,a) extended fractional differentiable for some $\alpha\in(0,1)$ on (c,d)

  then we have

  I. if $D_a^{\alpha}f(t)>0\,\text{for all}\,\tau\in(c,d)\,\text{iff}\,1)\,a>1\,\text{and}\,\frac{df(t)}{dt}>0$ and $a\in(0,1]\,\text{and}\,\frac{df(t)}{dt}<0$

  if $D_a^{\alpha}f(t)<0\,\text{for all}\,\tau\in(c,d)\,\text{iff}\,1)\,a\in(0,1]\,\text{and}\,\frac{df(t)}{dt}>0$

  $$\text{and}\,a>0\ \text{and}\,\frac{df(t)}{dt}<0$$

  Proof follows from following remark and by first order derivative.

  $$D_a^{\alpha}f(t)=(\ln a)t^{1-\alpha}\frac{df(t)}{dt}.$$

- *Theorem 3.12 :*
  Let $a,c>0$ and $f,:[c,d]\to R$

  be a given function that holds the following

  f is continuous in $[c,d]$,

  f is (α,a) extended fractional differentiable for some $\alpha\in(0,1)$ on (c,d)

  if $D_a^{\alpha}f(t)=0\,\text{for all}\,t\in(c,d)$ then $\varphi(\tau)=C,C\in R$

  Proof: by using the same argument in mean value theorem 2.9 $\exists e\in(c,d)$ with

$$f_a^{\alpha}(e)=\frac{f(d)-f(c)}{\frac{d^{\alpha}}{\alpha(\ln a)}-\frac{c^{\alpha}}{\alpha(\ln a)}}=0$$

$$0=f(d)-f(c)$$

$$f(d)=f(c)$$

but a,b are arbitrary in $[c,d]$ with c<d therefore $f(\tau)=C,C\in R$.

Theorem 3.13 : Let $a,c>0$ and $f,g:[c,d]\to R$

be (α,a) extended fractional differentiable for some $\alpha\in(0,1)$ with $f_a^{\alpha}(t)=g_a^{\alpha}(t),\text{for all}\,t\in(c,d)$ then there exist $c\in R$ such that

$$f(t)=g(t)+c$$

Proof: Consider $\chi(\tau) = f(t) - g(t)$ then

$$\chi_a^{\alpha}(t) = 0 \text{ for all } t \in (c,d)$$

Then by theorem 2.12 we have

$$f(t) = g(t) + c$$

Extended fractional integral:

In this section we define the inverse operation of extended fractional derivative i.e Extended fractional integral.

Definition:

(α, a) Extended fractional integral:

let α ∈ (0, 1] and $t > 0, a > 0, a \in R$ let f be a function defined on (0, ∞] then the (α, a) Extended fractional integral is defined by

$$I_a^{\alpha} f(t) = \int_0^t \frac{f(s)}{(\ln a) s^{1-\alpha}} ds$$

Provided the integral exist

- *Remark 4.1:*

  It can be observed that the definition 4.1 turns out to be the classical definition of the integration of first order for the special case α = 1 and a = e.

  Remark 4.2 Setting a = e in the definition 4.1, we get the definition of the Conformable Fractional integral as defined by R. Khalil *et al* in 2014.

  Theorem 4.1 (Inverse Property) If f:[0,∞)→ R is any continuous function in the domain of I^{α} and 0 < α ≤ 1 $a > 0, a \in R, b > 0$ then for t > 0

$$D_a^{\alpha} I_a^{\alpha} f(t) = f(t)$$

Proof:

since f is continuous then is I_a^{α} (α, a) extended fractional differentiable

$$D_a^{\alpha} I_a^{\alpha} f(t) = (\ln a) t^{(1-\alpha)} \frac{d f(t)}{dt} \int_0^t \frac{f(s)}{(\ln a) s^{(1-\alpha)}} ds$$

$$= (\ln a) t^{(1-\alpha)} \frac{f(t)}{(\ln a) t^{(1-\alpha)}} = f(t)$$

- *Theorem 4.2 : Fundamental theorem of Calculus for Extended fractional derivative :*

if f: [0, ∞) → R is any continuous function in the domain of I_a^{α} and 0 < α ≤ 1, $a > 0, a \in R, a > 0$ then for $t > 0$

$$I_a^{\alpha} D_a^{\alpha} f(t) = f(t) - f(0).$$

$$I_a^{\alpha} D_a^{\alpha} f(t) = f(t) - f(0).$$

$$I_a^{\alpha} \mathbf{D}_a^{\alpha} f(t) = \int_0^t \frac{t^{(1-\alpha)} \frac{d f(t)}{dt}}{(\ln a) s^{(1-\alpha)}} (\ln a)$$

$$I_a^{\alpha} \mathbf{D}_a^{\alpha} f(t) = \int_0^t \frac{t^{(1-\alpha)} \frac{d f(t)}{dt}}{(\ln a) t^{(1-\alpha)}} (\ln a)$$

$$I_a^{\alpha} \mathbf{D}_a^{\alpha} f(t) = \int_0^t \frac{d f(t)}{dt}$$

$$I_a^{\alpha} \mathbf{D}_a^{\alpha} f(t) = f(t) - f(0).$$

5. Applications

Example 1: Extended fractional differential equation

$$D_{\frac{1}{2}}^{\alpha} f(t) = (1+t)^n, f(0) = 0$$

$$\left(\ln \frac{1}{2}\right) t^{\frac{1}{2}} \frac{d f(t)}{dt} = (1+\boldsymbol{t})^{\boldsymbol{n}}$$

Expanding into maclaurin series

$$\left(\ln \frac{1}{2}\right) t^{\frac{1}{2}} \frac{d f(t)}{dt} = \sum_{k=0}^{\infty} \binom{n}{k} t^k, \quad f(0) = 0$$

$$\frac{d f(t)}{dt} = \frac{1}{\left(\ln \frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n}{k} t^{k-\frac{1}{2}}, \quad f(0) = 0$$

$$\int d f(t) = \frac{1}{\left(\ln \frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n}{k} \int t^{k-\frac{1}{2}} dt, \quad f(0) = 0$$

$$f(t) = \frac{1}{\left(\ln \frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{t^{k+\frac{1}{2}}}{k+\frac{1}{2}} + c$$

Applying $f(0)=0$ we get

$$f(t)=\frac{1}{\left(\ln\frac{1}{2}\right)}\sum_{k=0}^{\infty}\binom{n}{k}\frac{t^{k+\frac{1}{2}}}{k+\frac{1}{2}}$$

Example 2. Extended fractional differential equation:

$$D_{\frac{1}{2}}^{\alpha}f(t)=t^2\cos(t),f(0)=0$$

$$\left(\ln\frac{1}{2}\right)t^{\frac{1}{2}}\frac{df(t)}{dt}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2(k+1)},\quad f(0)=0$$

$$\frac{df(t)}{dt}=\frac{1}{\left(\ln\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+\frac{3}{2}},\quad f(0)=0$$

$$\int_0^{\infty}df(t)=\int_0^{\infty}\frac{1}{\left(\ln\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+\frac{3}{2}}dt$$

$$f(t)=\frac{1}{\left(\ln\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\left(2k+\frac{5}{2}\right)}t^{2k+\frac{5}{2}}+c$$

and $f(0)=0$

$$f(t)=\frac{1}{\left(\ln\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\left(2k+\frac{5}{2}\right)}t^{2k+\frac{5}{2}}$$

Conclusions:

we have defined a new fractional derivative, called the Extended Fractional Derivative, which depends upon two parameters.

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