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RESEARCH ARTICLE

Existence and uniqueness of solutions for exponential fractional differential equations

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Abstract

In this paper, we have discussed some results on existence and uniqueness theorems for exponential definition fractional differential equation of order $\alpha,\ 0<\alpha<1$ And proved the theorem existence and uniqueness theorems for sequential exponential definition fractional differential equations of order n $\alpha, n<\alpha\leq n+1$ and $0<n\alpha<1$ and give examples to support results.

Keywords: Conformable fractional derivative, exponential fractional derivative, existence and uniqueness theorems, sequential exponential fractional differential equations.

MSC2020-Mathematics Subject Classification: 34B15

Introduction

Due to the applicability of the derivatives and integrals of the fractional order with the corresponding initial and boundary conditions, the theory of fractional calculus has drawn the attention of many researchers following the broad and successful applicability of the theory of differential equations in the fields of Applied Mathematics, Mathematical Physics, Chemical Sciences, Biological Sciences, Engineering and Technology, etc. In addition to the previously listed disciplines and technologies, the theory of fractional calculus is used in the fields of fluid dynamics, electromagnetism, viscoelasticity, feedback amplifier and capacitor analysis, etc. Numerous scholars have noted in recent decades that fractional order differentials and integrals are particularly crucial for explaining the viscoelastic characteristics of the real materials like polymers. For numerous centuries, the derivative of non-integer order has been a fascinating area of study. Different definitions of fractional derivative such as Riemann-Liouville fractional derivative , Caputo fractional derivative are a few of the fractional derivative types in fractional calculus. Fractional integrals are used to de ne the majority of fractional derivatives. The same rationale gives those fractional derivatives various non-local behaviours, which opens up a wide range of intriguing applications, such as future dependency and memory effects.

The majority of the fractional derivatives are defined by means of the fractional integral. All these fractional derivatives do not satisfy the properties of the classical integer order derivatives. We are aware that the integer order derivative of a constant is zero and we expect the same in case of fractional derivatives, but this is not the case in most of the fractional derivatives except the Caputo fractional derivative. Also the properties of the classical derivatives like the Product rule, the Quotient rule, the Chain rule, Rolle's and Lagrange's Mean value theorems are not satisfies by fractional derivatives(Podlubny, I(1999)., Miller, K. S. and Ross, B(1993)). Few new fractional derivatives, called them the conformable fractional derivatives and proved some of the properties that are not satisfied by the earlier fractional derivatives. All these conformable derivatives are some extensions of the classical limit form definition (Katugampola U. N. 2014 , Khalil R.2014 , Abdljawad et al. 2015 , Hammad M. et al. 2014 , Almeida, R. et al 2017.)

Methods and Materials

Exponential Fractional Derivative

In this paper we have given an another natural extension of the limit form derivative and we call it the exponential

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fractional derivative(kamble R.et al. 2024). Proved the existence of the solution of exponential fractional initial value problem and sequentially homogeneous exponential fractional initial value problem. First we define the exponential fractional derivative as follows.

Defination

Given a function : γ [0,∞) --→ R. Then the Exponential fractional derivative'' of γ of order α $\in (0,1]$, $a > 1$, $a \in R$ is defined by

$$D^{\alpha}\gamma(\tau) = (\gamma)^{\alpha}(\tau) = T_{\alpha}\gamma(t) = \lim_{\mu\to 0}\frac{\gamma(\tau + \mu a^{(\alpha-1)\tau}) - \gamma(\tau)}{\mu} \quad \ldots(1)$$

for all $\tau > 0$, Provided the limit exist.

α ∈(0, 1). In this case, we say that ϕ is α exponential fractional differentiable at τ.

If γ is α-differentiable in some (0, a), a > 0, and $\lim_{\tau\to 0^+}\gamma^{\alpha}(\tau)$ exists then define $\lim_{\tau\to 0^+}\gamma^{\alpha}(\tau) = \gamma^{\alpha}(0)$.

Remark

If let γ be α, exponential, $a > 1, a \in R$ Exponential fractional diffferentiable function , $\alpha \in (0,1]$ *then*

$$T_{\alpha}(\gamma)(\tau) = a^{(\alpha-1)\tau}\frac{d\gamma(\tau)}{d\tau} \quad \ldots(2)$$

This definition follows classical results like the product rule, the quotient rule. The chain rule and the results which are similar to Rolle's theorem and the mean value theorems and obtained the exponential fractional derivative of some of the elementary functions like the exponential function, polynomial function and 2 trigonometric functions. and define exponential fractional derivative of higher order as follows

Definition

let α ∈ (n, n + 1], $a > 1, a \in R$, n ∈ N, a>1 , γ be a function n differentiable for τ > 0 then $\gamma: (0,\infty) \to R$

$$D^{\alpha}(\gamma)(\tau) = \lim_{\mu\to 0}\frac{\gamma^{n}(\tau + \mu a^{(\alpha-\lceil\alpha\rceil)\tau}) - \gamma^{n}(\tau)}{\mu}$$

Where $\lceil\alpha\rceil = n + 1$ is the smallest integer greater than or equal to α

if γ is α- comformable differentiable in some (0, b), b >0 and $\lim_{\tau\to 0^+}\gamma^{\alpha}(\tau)$,exists then define $\lim_{\tau\to 0^+}\gamma^{\alpha}(\tau) = \gamma^{\alpha}(0)$.

Remark

$$T_{\alpha}(\gamma)(\tau) = a^{(\alpha-\lceil\alpha\rceil)\tau}\frac{d^{n+1}\gamma(\tau)}{(d\tau)^{n+1}}$$

where we assume that n < α ≤ n+1,n ∈ N and γ is (n+1) times α- exponential di erentiable at τ > 0, $\lceil\alpha\rceil = n + 1$ is the smallest integer greater than or equal to α.

We can continue our definition of exponential fractional integration in a natural way based on the classical integration

Definition

let α ∈ (0, 1] and $a > 1, a \in R$, let γ be a function defined on (0, τ] then the α exponential fractional integral

$$I^{\alpha}\gamma(\tau) = \int_0^{\tau}\frac{\gamma(s)}{a^{(\alpha-1)s}}ds \quad \ldots(3)$$

where τ ≥ 0 provide the integral exist.

Definition

let α ∈ (0, 1] and,$a > 1, a \in R$ b > 0 let γ be a function defined on (b, τ] then the α exponential fractional integral

$$I_b{}^{\alpha}\gamma(\tau) = \int_0^{\tau}\frac{\gamma(s)}{a^{(\alpha-1)s}}ds \quad \ldots(4)$$

where τ ≥ 0 provide the integral exist(kamble R.et al. 2024).

Now we have enough material to prove our main results in the following sections

Observation\Results

Existence and uniqueness theorem on Exponential **defination fractional** *differential equations*

Existence of Fractional Neural Functional Difference Equations is proved (Agarwal, R. P., Zhou, Y., Yunyun 2010).

Existence Results for Sequential Fractional Integro-differential Equations with Nonlocal Multi-point and Strip Conditions is proved(Ahmad, B., Ntouyas, S. K., Agarwal, R. P. and Alsaedi, A.2016).

Existence of Solutions for Non-linear Fractional Integro-differential Equations is proved (Bragdi, A., Frioui, A. and Guezane L. A.2020).

Mild Solutions for Fractional Differential Equations with Nonlocal Conditions is proved (Fang Li.2010).

Existence of Mild Solutions for an Impulsive Fractional Integro-di fferential Equations with Non-Local Condition is proved (Hilal, K. , Ibnelazyz, L. , Guida, K. and Melliani, D.2019).

On Some Existence and Uniqueness Results for Non linear Fractional Differential Equations with Boundary Conditions is is proved (Kamble, R. M. and Kulkarni, P. R.2023).

New Existence and Uniqueness Results for Non linear Fractional Differential Equations With Composite Boundary Conditions is proved (Kamble, R. M. and Kulkarni, P. R.2023)

Solution of Fractional Order Differential-Difference Equation by Using Laplace Transform Method is proved (Kamble, R. M. and Kulkarni, P. R.2024) New Exponential Definition of fractional derivative is introduced (Kamble, R. M. and Kulkarni, P. R.2024)

Abels formula and Wronskian for conformable fractional differential equations is evaluated(Hammad M.and R. Khalil.2014)

Basic theory of differential equations is given in 'A second course in Elementary Ordinary Differential Equations' (Finan M 2013).

Existence and Uniqueness of Solutions for Non-linear Fractional Integro-differential Equations with Non-local Boundary Conditions is proved (Mardanov, M. J., Sharifov, Y. A. and Aliyev, H. N.2022).

Solvability for a system of Non-linear Fractional Higher-order Three Point Boundary Value Problem is proved(Rao, S. N.,2017)

Remark on Local Fractional Calculus and Ordinary Derivatives Is given (Almeida, M. Guzowska and T. Odzijewicz A.,2017).

A new definition of fractional derivative is introduced known as conformable fractional derivative (Khalil R., Horani M. Al, Yousef A., and Sababheh M.,2014)

Remarks On conformable fractional calculus is given (Abdeljawad T.,2015)

A new fractional derivative with classical properties is proved similar to conformable fractional derivative (Katugampola U. N.,2014)

Considering the different initial and boundary conditions and by the applications of a variety of fixed point theorems, many authors have proved the existence and uniqueness results. The authors have examined a boundary value problem with boundary conditions of the type x(0) = x(1) = 0 involving Caputo non linear fractional integro-differential equations of order $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. For this, the authors have used the fixed point theory to prove a number of existence and uniqueness conclusions. Specially, Krasnoselskii's fixed point theorem, under certain weak conditions, and the Banach contraction mapping principle is used. A few examples are provided to bolster the findings. The authors have proved the existence and uniqueness of solutions to non-linear fractional integro-differential equations with composite conditions where the fractional differential operator under consideration is in the sense of Caputo. By means of the Laplace transform, the authors have obtained the solution of fractional order differential-difference equation. In this section, we have proved the existence and uniqueness of solutions to exponential fractional differential equations with certain initial conditions.

In all the paper any Exponential fractional derivative lies in (0,1)

Theorem

Suppose

$p(\tau), Q(\tau)$ are continous functions on $I = [a,b]$ and γ be $\alpha -$ Exponential fractional differentiable with $\alpha \in (0,1], \tau_0 \in I$ then the solution of initial value problem(I.V.P)

$$D^{\alpha}\gamma + P(\tau)\gamma = Q(\tau) \quad \ldots(5)$$

$$\gamma(\tau_0) = \gamma_0 \quad \ldots(6)$$

Has unique solution on I, $\tau_0 \in I$

Proof: The exponential fractional derivative of γ at τ is given as follows

$$D^{\alpha}\gamma(\tau) = (\gamma)^{\alpha}(\tau) = T_{\alpha}\gamma(t) = \lim_{\mu \to 0} \frac{\gamma(\tau + \mu a^{(\alpha-1)\tau}) - \gamma(\tau)}{\mu}, \text{ for } \alpha \in (0,1] \quad \ldots(7)$$

provided the limit exists. It can be verified that

$$T_{\alpha}(\gamma)(\tau) = a^{(\alpha-1)\tau}\frac{d\gamma(\tau)}{d\tau} \quad \ldots(8)$$

Hence the differential equation (5)can be expressed as

$$a^{(\alpha-1)\tau}\frac{d\gamma(\tau)}{d\tau} + p(\tau)(\gamma)(\tau) = q(\tau)$$

$$\frac{d\gamma(\tau)}{d\tau} + a^{(1-\alpha)\tau}P(\tau)(\gamma)(\tau) = a^{(1-\alpha)\tau}q(\tau) \quad \ldots(9)$$

Since P(τ),Q(τ) are continuous functions defined on the interval I = [a,b],a,b ∈ R,a > 1 and $a^{(\alpha-1)\tau}$ is continuous, it follows from the existence and uniqueness theorem for ordinary differential equations that the initial value problem given by equations (9) and (6) has a unique solution. This proves that the initial value problem (5) (6) has a unique solution. The unique And the unique solution is given by

$$\frac{d\gamma(\tau)}{d\tau} = a^{(1-\alpha)\tau}[Q(\tau) - P(\tau)(\gamma)(\tau)]$$

$$\gamma(\tau) = \gamma(\tau_0) + \int_{\tau_0}^{\tau} a^{(1-\alpha)\tau}[Q(\tau) - P(\tau)(\gamma)(\tau)]d\tau$$

Theorem

P_(n − 1) (τ),P_(n − 2) (τ),… P_1 (τ),P_0 (τ)are continous functions on I = [a, b]

$n\alpha \in (0,1]$.

then the solution offollowing initial value problem

$$D^{n\alpha}\gamma + P_{n-1}(\tau)D^{(n-1)\alpha})\gamma \ldots + P_2(\tau)(D^{2\alpha})\gamma + P_1(\tau)D^{\alpha}\gamma + P_0(\tau)\gamma = Q(\tau) \ldots \quad \ldots(10)$$

$$\gamma(\tau_0) = \gamma_0, \; D^{\alpha}(\gamma)(\tau_0) = \gamma_{1,\ldots,} D^{(n-1)\alpha})(\gamma)(\tau_0) = \gamma_{n-1} \qquad t_0 \in I = (a,b), a \geq 0 \quad \ldots(11)$$

Possesses unique solution

Proof

let

$\mu_1 = \gamma(\tau)$, $\mu_2 = D^{\alpha}\gamma(\tau)$, $\mu_3 = D^{2\alpha}\gamma(\tau)$, $,\mu_n = D^{(n-1)\alpha}\gamma(\tau)$,

here μ_1 *is the solution* .

Therefore we have

$$D^{\alpha}\mu_1 = \mu_2$$

$$D^{\alpha}\mu_2 = \mu_3 \ldots$$

$$D^{\alpha}\mu_{n-1} = \mu_n$$

Also, from the equation(10)

$$D^{\alpha}\mu_n = -P_{n-1}(\tau)\mu_n - \cdots - P_2(\tau)\mu_3 - P_1(\tau)\mu_2 - P_0(\tau)\mu_1 + Q(\tau)$$

The Above system can be written as

$$D^{\alpha}\begin{bmatrix}\mu_1\\ \mu_2\\ .\\ .\\ .\\ \mu_{n-1}\\ \mu_n\end{bmatrix} + \begin{bmatrix}0, & -1, & 0 & 0\ldots0\\ 0 & ,0,-1 & ,0,\ldots0 & \\ & . & & \\ & . & & \\ & . & & \\ 0 & ,0, & 0, & 0,\ldots\ldots-1\\ p_0, & p_1, & p_2, & p_3,\ldots.p_{n-1},\end{bmatrix}\begin{bmatrix}\mu_1\\ \mu_2\\ .\\ .\\ .\\ \mu_{n-1}\\ \mu_n\end{bmatrix} = \begin{bmatrix}0\\ 0\\ .\\ .\\ .\\ 0\\ q(\tau)\end{bmatrix}$$

$$D^{\alpha}U(\tau) + R(\tau)U(\tau) = S(\tau)$$

where

$$U(\tau) = \begin{bmatrix}\mu_1\\ \mu_2\\ .\\ .\\ .\\ \mu_{n-1}\\ \mu_n\end{bmatrix}, R(\tau) = \begin{bmatrix}0, & -1, & 0 & 0\ldots0\\ 0 & ,0,-1 & ,0,\ldots0 & \\ & . & & \\ & . & & \\ & . & & \\ 0 & ,0, & 0, & 0,\ldots-1\\ p_0, & p_1, & p_2, & p_3,\ldots p_{n-1},\end{bmatrix}, S(\tau) = \begin{bmatrix}0\\ 0\\ .\\ .\\ .\\ 0\\ q(\tau)\end{bmatrix},$$

$$\Gamma(\tau_0) = \begin{bmatrix} \gamma(\tau_0) = \mu_1(\tau_0) \\ D^{\alpha}(\gamma)(\tau_0) = \mu_2(\tau_0) \\ . \\ . \\ . \\ D^{(n-2)\alpha})(\gamma)(\tau_0) = \mu_{n-1}(\tau_0) \\ D^{(n-1)\alpha})(\gamma)(\tau_0) = \mu_n(\tau_0) \end{bmatrix} = \Gamma(\tau) \begin{bmatrix} \mu_1(\tau_0) = \gamma_0 \\ \mu_2(\tau_0) = \gamma_1 \\ . \\ . \\ . \\ \mu_{n-1}(\tau_0) = \gamma_{n-2} \\ \mu_n(\tau_0) = \gamma_{n-1} \end{bmatrix}$$

$U(\tau)' + a^{(1-\alpha)\tau}[R(\tau)U(\tau)] = a^{(1-\alpha)\tau}S(t)$ …(12)

And the initial condition changed to $\Gamma(\tau_0) = \Gamma(\tau)$ …(13)

By theorem in Finan M., A second course in Elementary Ordinary Differential Equations, it follows that the initial value problem (12) (13) has a unique solution. Hence the initial value problem (10) (11)has a unique solution.

Example 1

As an application of the results proved so far,we will solve an initial value problem. $D^{0.5}\gamma(\tau) + a^{\frac{-\tau}{2}}log a\gamma(\tau) = \tau a^{-\tau}$ with the initial conditions

$\gamma(0) = 0$

Proof: First we note that

$T_\alpha(\gamma)(\tau) = a^{(\alpha-1)\tau}\frac{d\gamma(\tau)}{d\tau}$

$D^{\alpha}(a^{\lambda\tau}) = \lambda a^{\lambda\tau}a^{(\alpha-1)\tau}log a$

Since the Exponential fractional derivative satisfies the product rule. We can apply to differential equation to solve differential equations.

$D^{0.5}(\gamma(\tau)a^{\tau}) = \tau$

$(\gamma(\tau)a^{\tau}) = \int_0^{\tau} a^{\frac{\tau}{2}}\tau d\tau + c$ (since by using definition of Exponential fractional integral)

After applying intial condition we get value of c and we get the unique Solution.

Or

By theorem 3.1 there exist unique solution and the unique solution is given by

$$\frac{d\gamma(\tau)}{dt} = a^{0.5}[\tau a^{-\tau} - a^{\frac{-\tau}{2}}log a\gamma(\tau)]d\tau$$

$\gamma(\tau) - \gamma(0) = \int_0^{\tau} a^{0.5}\left[\tau a^{-\tau} - a^{\frac{-\tau}{2}}log a\gamma(\tau)\right]d\tau$ (applying the initial condition)

$$\gamma(\tau) = \int_0^{\tau} a^{0.5}\left[\tau a^{-\tau} - a^{\frac{-\tau}{2}}log a\gamma(\tau)\right]d\tau$$

we get the unique solution.

Example 2

$D^{0.5}\gamma(\tau) + e^{\frac{-\tau}{2}}\gamma(\tau) = \tau e^{-\tau}$

$\gamma(0) = 0$

Proof: $D^{0.5}(\gamma a^{\tau}) = \tau$ (since by theorem 3 and using the product rule for Exponential fractiaonal derivative.)

$(\gamma e^{\tau}) = \int_0^{\tau} e^{\frac{\tau}{2}}\tau d\tau$ +c (since by using definition of Exponential fractional integral)

$$(\gamma e^{\tau}) = \int_0^{\tau} e^{\frac{\tau}{2}}\tau d\tau + c$$

$$\gamma(\tau) = (2\tau - 4)e^{\frac{-\tau}{2}} + ce^{-\tau}$$

after applying the initial condition, $\gamma(0) = 0$

c=4

$$\gamma(\tau) = (2\tau - 4)e^{\frac{-\tau}{2}} + 4e^{-\tau}$$

Is the unique solution.

By theorem 3.1 there exist unique solution and the unique solution is given by

$$\frac{d\gamma}{dt} = e^{0.5}[\tau e^{-\tau} - e^{\frac{-\tau}{2}}\gamma(\tau)]d\tau$$

$$\gamma(\tau) - \gamma(0) = \int_0^{\tau} e^{0.5}\left[\tau e^{-\tau} - e^{\frac{-\tau}{2}}\gamma(\tau)\right]d\tau$$

$$\gamma(\tau) = \int_0^{\tau} e^{0.5}\left[\tau e^{-\tau} - e^{\frac{-\tau}{2}}\gamma(\tau)\right]d\tau$$

is the unique solution.

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