The Scientific Temper (2024) Vol. 15 (3): 2470-2472
E-ISSN: 2231-6396, ISSN: 0976-8653
Doi: 10.58414/SCIENTIFICTEMPER.2024.15.3.09
https://scientifictemper.com/
RESEARCH ARTICLE

# Existence of a homeomorphism from the space of continuous functions to the space of compact Subsets of a topological space, X

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## Abstract

This paper presents proof that there exists a subspace of the space of continuous functions on a topological space X, which is homeomorphic to the space of compact subsets of X. Those let C(X) denote the space of continuous functions on a topological space X and K(X) be the space of compact subsets of X. We prove that there exists a subspace of C(K(X)) which is homeomorphic to C(X). The result remains valid for compact open topology and point-wise convergence topology on K(X).

**Keywords**: Hyperspace, Regular, Compact-open topology, Point-wise convergence topology.

## Introduction

This paper establishes a topological equivalence between two seemingly disparate spaces: the space of continuous functions on a topological space *X* and the space of compact subsets of *X*. We demonstrate that a specific subspace of continuous functions can be directly correlated with the space of compact sets through a homeomorphism. This unexpected connection provides new insights into the structural properties of these spaces and opens avenues for further exploration in topology and analysis (Michael, 1951).

Suppose *X* is a Hausdorff space and *K*(*X*) be the hyperspace of compact subsets of *X* with Vietoris topology (Charatonik, 1998). A sub-base for Vietoris topology is given by

$$\langle Ui \rangle = \{ K \in K(X) : K \subseteq \bigcup Ui, K \cap Ui = \emptyset, \forall i \}$$

where *Ui* is a finite collection of open subsets of *X* (Michael, 1951). Let *C*(*X*) and *C*(*K*(*X*)) denote space of continuous functions on *X* and *K*(*X*), respectively. We consider two topologies on *C*(*X*) and *C*(*K*(*X*)).In Section 1, *C*(*X*) and *C*(*K*(*X*)) are endowed with compact open topology for which a sub-basic set for *C*(*X*) is of the form (Michael, 1951; Hosokawa, 1997).

$\langle K : U \rangle = f \in C(X) : f(K) \subseteq U$ where $K \subseteq X$ is compact and $U \subseteq X$ is open.For *C*(*K*(*X*)) sub basic set is of the form

$\langle \tilde{K} : \tilde{U} \rangle = \{ \tilde{f} \in C(K(X)) : \tilde{f}(K) \subseteq U \}$ where $\tilde{K} \subseteq K(X)$ is compact and

$U\ K(X)$ is open. In Section 2, we consider *C*(*X*) and *C*(*K*(*X*)) with point-wise convergence topology for which sub base for *C*(*X*) and *C*(*K*(*X*)) are

$\langle p : U \rangle = f \in C(X) : f(p) \in U$ where $p \in X$ and $U \subseteq X$ is open in *X* and

$\langle P : \tilde{U} \rangle = \{ \tilde{f} \in C(K(X)) : \tilde{f}(P) \in \tilde{U} \}$ where $P \in K(X)$ and $\tilde{U} \subseteq K(X)$ is open

in *K*(*X*) (Mizokami, 1998).

Let $f : X \to X$ be continuous, then *f* induces a continuous function on *K*(*X*),

$\tilde{f} : K(X) \to K(X)$ defined as $\tilde{f}(K) = f(K), \forall K \in K(X)$. We prove the continuity of the function $\phi : C(X)$ $C(K(X))$ defined as $\phi(f) = f$ (Hosokawa, 1997).

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**How to cite this article:** Amalraj, P.,Vinodkumar, P. B. (2024). Existence of a homeomorphism from the space of continuous functions to the space of compact Subsets of a topological space, X. The Scientific Temper, **15**(3):2470-2472.

Doi: 10.58414/SCIENTIFICTEMPER.2024.15.3.09

**Source of support:** Nil

**Conflict of interest:** None.



**Received:** 15/07/2024 **Accepted:** 09/08/2024 **Published** : 16/09/2024

In (2),it is proved that if $X$ is a compact (Nadler, 1978), connected metric space and if $f$ is a homeomorphism, then $\phi(f)$ is a homeomorphism. In (5), it has shown that not all maps on $K(X)$ are inducible.But the first result about $\phi$ is given in (4) (Michael, 1951), where it is proved that $C(X)$ can be embedded in $C(K(X))$ with a compact open topology on $C(X)$ and point-wise convergence topology on $C(K(X))$.In this paper, we discuss the continuity of $\phi$ with the same topology on $C(X)$ and $C(K(X))$. A detailed discussion on $K(X)$ an be found in (1),(3) (Michael, 1951; Nadler, 1978).

### *Compact Open Topology On C(X) and C(K(X))*

Throughout this section, $C(X)$ and $C(K(X))$ are endowed with compact-open topology.Let $\phi : C(X) \to C(K(X))$ be defined as $\phi(f) = \tilde{f}$ where $\tilde{f}(K) = f(K)$ $\forall K \in K(X)$. We ask four simple questions about $\phi$.

- Is $\phi$ continuous?
- Is $\phi$ on-one ?
- Is $\phi$ on to ?
- Is $\phi$ is a homeomorphism?

Motivated by Theorem 2(4) (Hosokawa, 1997), we first give an example to show that $\phi$ is not on to.

**Example 1.1:** Let $\tilde{f} : K(\mathbb{R}) \to K(\mathbb{R})$ be defined as $\tilde{f}(K) = \{Inf(K)\}$, $\forall K \in K(\mathbb{R})$ For example,$\tilde{f}([3, 6]) = \{3\}$ and $\tilde{f}([4, 5]) = \{4\}$

note that,$[4, 5] \subseteq [3, 6]$ but $\tilde{f}[4, 5] \not\subset \tilde{f}([3, 6])$

If $\tilde{f}$ is an induced map,then $A \subset B$ implies $\tilde{f}(A) \subset \tilde{f}(B)$ in (4). Therefore, $f$ is not an induced map.

that is, $\tilde{f} \notin \phi(C(\mathbb{R}))$

Hence answer to question number (3) is No.

The following result is trivial, but we give the proof for the sake of completeness.

**Proposition 1:** *$\phi$ is one-one*

*Proof.* $\phi(f1) = \phi(f2)$ implies $f1(K) = f2(K)$, $\forall K \in K(X)$ $f1(\{x\}) = f2(\{x\})$, $\forall \{x\} \in K(X)$

$f1(x) = f2(x)$, $x$ $X$

$f1 = f2$. □

Question number(2) is answered positively.

Next we prove the continuity of $\phi$.

**Definition 1.2:** A topological space $X$ is a regular space iff whenever $A$ is closed in $X$ and $x \notin A$,then there are disjoint open sets $U$ and $V$ with $x \in U$ and $A \subset V$ (Hurewicz, n.d. ; Kluge, n.d.)

**Proposition 2:** *If X is regular, then $\phi$ is continuous.*

To prove this proposition, we need a result proved in (1) (Michael, 1951), which we quote as a lemma given below.

**Lemma 1.3:** *(1) (Michael, 1951)Let X be regular. If $\tilde{K} \subseteq K(X)$, then*

$K \in \tilde{K}$

$K \in K(X)$

Proof of Proposition 2

Let $V$ be a sub-basis neighbourhood of $\phi(f)$. Then there exist sub basic neighbour- hood $Ui$ of $f$ such that $\phi(Ui)V$ .4 $i$

Let $\tilde{K}$ be a compact subset of $K$, then by lemma 1.2, $K$ is compact. Hence

$K \in \tilde{K}$

sub basis open sets of $C(K(X))$ are given by Σ

$\tilde{K} : \langle U_i \rangle = \psi \in C(K(X)) : \psi(\tilde{K}) \subset \langle U_i \rangle$

where $\langle U_i \rangle = K \in K(X) : K \subseteq \bigcup_i U_i, K \cap U_i \neq \emptyset, \forall i$ .

ie, $\tilde{K} : \langle U_i \rangle = \psi \in C(K(X)) : \psi(K) \subset \bigcup_i U_i, \psi(K) \cap U_i \neq \emptyset, \forall K \in \tilde{K}, \forall i$

Let $\tilde{K} : \langle U_i \rangle$ be a sub basis neighbourhood of $\phi(f)$

$\phi^{-1}\ \tilde{K} : \langle U_i \rangle =$

$f \in C(X) : f\ \bigcup_{K \in K} K \subset \bigcup_i U_i, f(K) \cap U_i\ \emptyset, \forall K \in \tilde{K}, \forall i$

$= f \in C(X) : f\ \bigcup_{K \in K} K \subset \bigcup_i U_i\ \bigcap\ f \in C(X) : f(K) \cap\ \emptyset, \forall K \in \tilde{K}, \forall i$

$U_i =$

Σ. $K : \bigcup U_i\ \bigcap\ f \in C(X) : f(K) \cap\ \emptyset, \forall K \in \tilde{K}$

$U_i =$

$K \in \tilde{K}$ $i$ $i$

### *Claim*

For each $f-1(Ui)$ there is an open sub set $Vi$ of $X$ such that $Vi$ $f-1(Ui)$ and for each (Weiss & Kifer, n.d.; Lyubich, n.d.).

(1.1) $K \in \tilde{K}$, $K \cap Vi \neq \emptyset$

if not , for each $Vi$, there exist $KVi \in \tilde{K}$ such that $KVi \cap Vi = \emptyset$

Let $O(U) = \{U' : U' \subset U\}$.Then $(O(U), \subseteq)$ is a poset. Hence, $N : Vi \rightarrowtail KV$ is a

net.$N$ has a convergent cofinal sub net since $\tilde{K}$ this sub net (Saito, 2000).Then,

is compact.Let $K0$ be the limit of

$K0 \cap Vi = \emptyset$, $\forall Vi \subset Vi \subset f-1(Ui)$

Since $X$ is regular, $Vi = U$.

i

therefore $K0\ f-1(Ui) =$ . This is a contradiction to the fact that $f(K0)\ Ui =$ . Hence the claim holds:

Choose $Vi \subset f-1(Ui)$ with the property (1.1). Let $xi \in Vi \cap K$ for each $Vi$ and $K$.

Define $Ki = \{xi\}$. $Ki$ is compact as it is a closed subset of S $K$ which is compact.

$\therefore f(Ki) \subset V_i \subset Ui$. So $f \in \langle Ki : Ui \rangle$

$K \in \tilde{K}$ Σ

$\phi \langle Ki, Ui \rangle \subset$

∴ Σ. S

$f \in C(X) : f(K) \cap Ui \neq \emptyset, \forall K \in \tilde{K}$

: S $Ui$ΣΣ T ΣT .. $\tilde{K}\ i : \langle Ui \rangle$ΣΣ $\subset \phi-1$ .. $\tilde{K} : \langle Ui \rangle$ΣΣ

$\phi-1\ \tilde{K} : \langle Ui \rangle$ is an open set in $C(X)$ So $\phi$ is continuous (Hosokawa, 1997; Charatonik, 1998; Mizokami, 1998).

**Proposition 3:** *$\phi-1$ is continuous.*

*Proof.* Since $X$ is regular,$C(K(X))$ is regular. So $\phi(C(X))$ $C(K(X))$ is regular.

By proposition.1,$\phi$ is one-one.Hence by similar argument as in proposition.2, $\phi-1$ is continuous. □

***Corollary 1.3.1***

*If X is regular, then C(X) is embedded in C(K(X)).*

*Proof.* $\phi$ is a homeomorphism from $C(X)$ to $\phi(C(X))$ by proposition.2 and 3. ie, $C(X) \sim= \phi(C(X))$ $C(K(X))$.

This answers question number(4) (Wang, n.d.). □

Point-wise Convergence Topology On $C(X)$ and $C(K(X))$

In this section, we consider point-wise convergence topology on $C(X)$ and $C(K(X))$ to answer questions (1) to(4) (Saito, 2003).

Obviously, answers to questions (2) and (3) are same as given in proposition(1) and example(1.1), respectively.

**Proposition 4:** *$\phi$ is continuous.*

*Proof.* Let $\phi(f) \in \langle K : \langle Ui \rangle\rangle$ where $K \in K(X)$ and $Ui$ is an open set in $K(X)$

$f(K) \cap Ui /= \emptyset$ implies $f(xi) \in Ui \cap f(K)$ for some $xi \in K$

Then $\langle xi : Ui \rangle$ is a sub-basic neighbourhood of $f$, which means $\phi$ is continuous. □

**Proposition 5.** *$\phi$ is open from C(X)to $\phi(C(X))$* (Zhao, n.d.).

T$n$

*Proof.* Let

T$n$ $i$=1

$\langle xi : Ui \rangle$ be any basis open set in $C(X)$

Let $f \in$

Then,$\phi(f) \in$ $i$=1

$\langle xi : Ui \rangle$

T$n$ T $\langle \{x_i\} : \langle U_i \rangle\rangle$ $\phi(C(X))$which is open in $\phi(C(X))$ □

**Proposition 6.** *$C(X)$ is embedded in $C(K(X))$.*

*Proof.* $\phi$ is a homeomorphism from $C(X)$ to $\phi(C(X))$ $C(K(X))$ by proposition.4 and 5. □

## Acknowledgement

Authors thank the management of Rajagiri School of Engineering and Technol- ogy for providing research facilities.

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