Abstract
This paper investigates sum-perfect cube graphs, defined as graphs \( G = (V, E) \) with a bijection \( f : V(G) \rightarrow \{1, 2, ..., p - 1\} \). Where \( p \) is the number of vertices. For each edge \( uv \in E(G) \), a function \( f : E(G) \rightarrow N \) is defined by \( f(uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2f(v) \). If \( f \) is injective, \( f \) is termed a Sum perfect cube labeling. The study focuses on identifying graphs where all edges permit such a labeling, termed sum-perfect cube graphs. This paper explores the properties and classifications of these graphs.

Keywords: Sum perfect cube graphs, Graph labeling, Combinatorial mathematics, Number theory, Graph theory, Mathematical graphs, Cube labeling.

Introduction
Within the context of a newly established labeling scheme, the primary objective is to assign a perfect cube number to each edge of a graph. This study focuses on analyzing a finite, undirected graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges. Notations used include \( V(G) \) for the vertex set and \( E(G) \) for the edge set of graph \( G \). Standard graph theoretic notations are employed: \( P_n \) denotes a path with \( n \) vertices, \( C_n \) signifies a cycle with \( n \) vertices, \( K_n \) represents a star graph with \( n + 1 \) vertices, and \( T_{n,n} \) denotes a Tadpole graph. Terminology and notations follow established conventions, particularly those set by Harary.

Definition 1.1: A chord is an edge that connects two vertices of a cycle that are not contiguous.

Definition 1.2: We designate two chords in a cycle as twin chords if they combine to form a triangle with one of the cycle’s edges. The graph \( C_{n,p} \) comprises a cycle \( C_n \) with two twin chords and cycles \( C_p, C_{q} \), and \( C_{n+q} \) without chords created by the edges of \( C_n \). Furthermore, the graph \( C_{n,p} \) is for positive integers \( n \) and \( p \), where three is smaller than or equal to \( n \) minus 2.

Definition 1.3: graph of tadpoles Combining a vertex of the cycle graph \( C_m \) with one of the leaves of the path graph \( P_n \) results in the formation of a graph denoted by the notation \( T_{m,n} \).

Definition 1.4: Let \( G = (p, q) \) be a graph. A bijection \( f : V(G) \rightarrow \{1, 2, ..., p - 1\} \) is called sum perfect cube labeling of \( G \) if the induced function \( f : E(G) \rightarrow N \) by \( f(uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2f(v) \) is injective, for all \( u, v \in V(G) \). A graph that admits sum perfect cube (S.P.C.) labeling is called a sum perfect cube graph.

Literature Survey
In 2009, the investigation into Square Sum Graphs began, initiated by V. Ajitha et al. The authors demonstrated that certain graphs can be represented as square sums. Later, K. Germina and colleagues (2013) identified several other graphs as Square Sum graphs. The term “square difference graphs” was introduced by J. Shiama (2012), who also discovered graphs exhibiting square differences. S. Sonchhatra and G. Ghodasara (2016) defined a specific labeling known as Sum Perfect Square Labeling. They subsequently discovered various new Sum Perfect Square Graphs in 2016 and 2017, exploring different graph operations and graphs related to snakes. The concept of representing any number as a product of its prime factors’ powers allows identifying numbers as perfect cubes if the sum of their prime factors is a multiple of three. In this study,
a novel labeling scheme is developed to uniquely assign a perfect cube number to each edge of a graph. This labeling, termed sum-perfect cube labeling, characterizes graphs where every edge is labeled exclusively with perfect cube numbers, distinguishing them from others in the literature.

Main results

Lemma 3.1: If \( n \) is sum perfect cube graph, for all \( n \geq 2, n \in N \).

Proof: Let \( \{v_1, v_2, \ldots, v_n\} \) be the route \( P_n \) of \( n \) consecutive vertices. We define \( f:V(P_n) \to \{0,1,2,\ldots,n-1\} \) as \( f(v_i) = i-1 \leq i \leq n \).

Injectivity: When function \( f \) increases to \( i \), we get \( f(v_i) < f(v_{i+1}) \) and so \( f:E(G) \to N \). So \( f:E(G) \to N \) is injective. Hence \( P_n \) is the sum ideal cube graph for every \( n \in N, n \geq 2 \).

Lemma 3.2: Star \( K_{1,n} \) is the sum ideal cube graph for every \( n \in N \).

Proof: Let \( \{v_0, v_1, \ldots, v_n\} \) represent the succession of vertices of the star \( K_{1,n} \), where \( v_n \) is the apex vertex. We define \( f:V(K_{1,n}) \to \{0,1,2,\ldots,n\} \) as \( f(v_0) = 0 \) and \( f(v_i) = i, 1 \leq i \leq n \).

Injectivity: Because the value of function \( f \) is growing in terms of \( i \), \( f(v_i) < f(v_{i+1}) \) and so \( f:E(G) \to N \). So \( f:E(G) \to N \) is injective. Hence \( K_{1,n} \) is sum perfect cube graph, for all \( n \in N \).

Theorem 3.1: Every tree is a sum-perfect cube graph.

Proof: Let \( \{v_0, v_1, \ldots, v_n\} \) be the T-letter sequence of vertices in a tree. Permit us to imagine \( T \) as a root-bearing tree. Assume that \( v_0 \) represents the root vertex of \( T \). Let \( v_0 \) at vertex level 0. Let \( \{v_1, v_2, \ldots, v_n\} \) be the level's subsequent vertices, \( 1 \leq k \leq n \). \( v_{k+1}, v_{k+2}, \ldots, v_{n} \) be the successive vertices at level 2, \( k + 1 \leq t \leq n \) so on. We define \( f:V(T) \to \{0,1,2,\ldots,n\} \) by \( f(v_i) = i, 0 \leq i \leq n \).

Injectivity: At each consecutive level \( l_m \) and \( l_{m+1} \), \( m \in N \cup \{0\} \), we get \( f(v_i) < f(v_{i+1}) \) for all and \( i < m \), where vertex \( v_i \) is at level \( l_m \) and vertex \( v_j \) is at level \( l_{m+1} \). So \( f:E(T) \to N \) is injective. Hence, the tree is the perfect cube graph.

- A diagram displaying a tree with 15 vertices For a clearer understanding of the stated labeling scheme in Figure 1.

Theorem 3.2: \( C_n \) is the sum ideal cube graph for every \( n \geq 3 \).

Proof: Let \( \{v_1, v_2, \ldots, v_n\} \) be the successive vertices of the cycle \( C_n \). We define \( f:V(C_n) \to \{0,1,2,\ldots,n-1\} \) as follows:

\[
 f(v_i) = \begin{cases} 
 2i-2, & \text{if } 2 \leq i \leq n, \\
 2(n-1)+1, & \text{if } 0 \leq i \leq n-1.
\end{cases}
\]

Let \( f:E(G) \to N \) by \( f(\{u,v\}) = [f(u)]^2 + [f(v)]^2 + 3f(u)f(v) \) for all \( u, v \in E(C_n) \).

Injectivity

For \( 1 \leq i \leq \frac{n}{2} \) since \( f \) is increasing in terms of \( i \), \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \) and so we get \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \).

Moreover for \( \frac{n}{2} + 1 \leq i \leq n \), as \( f \) is decreasing in terms of \( i \), \( f(v_{i+1}) < f(v_i) < f(v_{i+2}) \) and so we get \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \).

Additionally, we see that \( f(v_1 v_n) = 1 \). It appears on the graph's edges the tiniest.

Additionally \( f(\{v_1,v_n\}) = \frac{n}{2} \) The graph's edge label is the highest. Right now \( f(v_i) \) is equal for all \( 1 \leq t \leq \frac{n}{2} \) and \( f(v_k) \) is odd for each \( k \), \( 1 \leq k \leq n - 1 \), we get \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \).

Also \( f(\{v_1,v_n\}) = \frac{n}{2} \). So \( f:E(G) \to N \) is injective. Hence \( C_n \) is the sum ideal cube graph for every \( n \geq 3, n \in N \).

Theorem 3.3: Cycle \( C_n \) is a perfect cube graph with one chord for all \( n \geq 4, n \in N \).

Proof: When \( G \) be the chord-only cycle. Let \( \{v_1, v_2, \ldots, v_n\} \) be the cycle's subsequent vertices \( C_n \) and \( e = v_1 v_n \) be the chord of the cycle \( C_n \). The vertices \( v_1, v_2, \ldots, v_n \) form a triangle in \( C_n \). With chord \( e \). We define \( f:V(C_n) \to \{0,1,2,\ldots,n-1\} \) as follows:

\[
 f(v_i) = \begin{cases} 
 2i-2, & \text{if } 2 \leq i \leq n, \\
 2(n-1)+1, & \text{if } 0 \leq i \leq n-1.
\end{cases}
\]

Let \( f:E(G) \to N \) by \( f(\{u,v\}) = [f(u)]^2 + [f(v)]^2 + 3f(u)f(v) \) for all \( u, v \in E(C_n) \).

Injectivity

The chord \( e = v_1 v_n \). It is labeled by 27, distinct from all other edge labels as it can only be induced by the vertex labels 0, 1, 2. However, the pair 0, 2 will never meet after labeling the vertex \( v_1 \) by one and \( v_n \) by 2. For \( 1 \leq i \leq \frac{n}{2} \), since \( f \) is increasing in terms of \( i \), \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \) and so we get \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \).

Moreover for \( \frac{n}{2} + 1 \leq i \leq n \), as \( f \) is decreasing in terms of \( i \), \( f(v_{i+1}) < f(v_i) < f(v_{i+2}) \) and so we get \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \)
Theorem 3.5: $K_n$ is a perfect cube graph with sums, for $n < 4$, $n \in \mathbb{N}$.

Proof: For $n = 1$ and $2$, according to Theorem 1, graphs are a subset of trees, and they are particular instances of trees, $K_1$ and $K_2$ are graphs of the sum perfect cube. For $n = 3$, $K_3$ is the cycle $C_3$, and hence, as per Theorem 2, $K_3$ is a perfect cube graph. For $n = 4$, the vertices of the graph are labeled when we encounter the pair 1, 2, and 0, 3, for which we have identical induced edge labels. This occurs when we are at the $K_4$ position. As a result, the sum perfect cube graph $K_4$ is not it. For everyone $n \geq 4$, as $K_4 \subseteq K_n \subseteq \ldots \subseteq K_m$, and so $K_n$ is not a perfect cube graph for all $n \geq 4$.

Theorem 3.6: Tadpole $T_{mn}$. For any positive values, the sum perfect cube graph is ideal integers $m$.

Proof: Let $V_{1, 2, \ldots, m}$ be the successive vertices of the cycle $C_m$ and let $V_{m, m+1, m+2, \ldots, m+n}$ be the successive vertices of a path $P_n$ in tadpole $T_{mn}$. Let $e = v_m, v_{m+1}$ be the bridge in tadpole $T_{mn}$, where $v_m$ It may be defined as the vertex with the highest label corresponding to the cycle $C_m$, and $v_{m+1}$ is the vertex with the smallest label corresponding to the path $P_n$.

We define $f : V(T_{mn}) \rightarrow \{0, 1, \ldots, m + n - 1\}$ as follows:

\[
f(v_i) = \begin{cases} 
m - 2i, & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, \\
2i - m - 1, & \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m \\
i - 1, & m + 1 \leq i \leq m + n
\end{cases}
\]

Injectivity

For $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$, since $f$ is decreasing in terms of $i$, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f \ast (v_i, v_{i+1}) > f \ast (v_{i+1}, v_{i+2})$. Moreover, for $\left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m$, as $f$ is decreasing in terms of $i$, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f \ast (v_i, v_{i+1}) > f \ast (v_{i+1}, v_{i+2})$.

Therefore, $f \ast (v_i, v_{i+1}) \neq f \ast (v_{i+1}, v_{i+2})$. Consequently, each edge label is unique. Hence $C_n$ a perfect cube graph with twin chords is available for all $n \geq 5$.

Corollary 1: The labeling pattern we specified in Theorems 3.2, 3.3, and 3.4 is justified by the information in Figure 2.

*Figure 2: Cycle, with one chord, twin chords in a perfect cube graph.*
Case 2: \( m \) is odd.

\[
f(v_i) = \begin{cases} 
2i - m - 1, & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, \\
2i - m - 1 \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m - 1' \\
i - 1, m + 1 \leq i \leq m + n
\end{cases}
\]

**Injectivity**

For \( 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \), since \( f \) is decreasing in terms of \( i \),

\[
f(v_i) > f(v_{i+1}) > f(v_{i+2})
\]

and so we get

\[
f^*(v_i, v_{i+1}) > f^*(v_{i+1}, v_{i+2}), 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1.
\]

For \( \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m + n \), as \( f \) is increasing in terms of \( i \),

\[
f(v_i) < f(v_{i+1}) < f(v_{i+2})
\]

and so we get

\[
f^*(v_i, v_{i+1}) < f^*(v_{i+1}, v_{i+2}), \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m + n.
\]

Moreover, for \( i \) is even, we get \( f^*(v_i, v_{i+1}) = 1 \) What is the edge label that is the smallest of all the edge labels in this graph? In addition, \( f^*(v_i, v_m) \) serves as the most prominent edge label in the cycle \( C_m \) As well as following the plan that we established, for \( m + 1 \leq i \leq m + n \), as \( f \) is increasing in terms of \( i \), \( f(v_i) < f(v_{i+1}) < f(v_{i+2}) \) and so we get

\[
f^*(v_i, v_{i+1}) < f^*(v_{i+1}, v_{i+2})
\]

Now \( f(v_t) \) is even for each \( t \), \( 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor \) and \( f(v_k) \) is odd for each \( k, \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq k \leq m \),

we get \( f^*(v_i, v_{i+1}) = f^*(v_k, v_{k+1}) \). This ensures that each of the edge labels is unique. As a result \( f^*: (E(T_{m,n})) \rightarrow N \) is injective in this case. Tadpole \( T_{m,n} \) is sum perfect cube graph for all positive integers \( m, n \).

According to the labeling pattern specified in the accompanying Figure 3, Theorem no. 3.6 is applicable.

**Conjecture**

If every vertex in an odd graph \( (G) \) has a weird degree, every vertex \( v \) is constructed so that \( d(v) \) is more significant than or equal to three. \( G \) is not a sum-perfect cube graph.

**Conclusion**

A graph’s edges can only be identified by a single, distinct, perfect cube integer. The labeling presented in this research will open up a new concept for studying particular families of graphs that include many graphs. Several new theorems have been shown in this study, all connected to the recently proposed labeling. When everything is said and done, one hypothesis has been proposed, which may be considered an open issue.

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**References**


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