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RESEARCH ARTICLE

Sum perfect cube labeling of graphs

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Abstract

This paper investigates sum-perfect cube graphs, defined as graphs $G = (V, E)$ with a bijection $f : V(G) \to \{0,1,2,\ldots,p-1\}$. Where p is the number of vertices. For each edge $uv \in E(G)$, a function $f*: E(G) \to N$ is defined by $f*(uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2 f(v)$. If $f*$ is injective, f is termed a Sum perfect cube labeling. The study focuses on identifying graphs where all edges permit such a labeling, termed sum-perfect cube graphs. This paper explores the properties and classifications of these graphs.

Keywords: Sum perfect cube graphs, Graph labeling, Combinatorial mathematics, Number theory, Graph theory, Mathematical graphs, Cube labeling.

Introduction

Within the context of a newly established labeling scheme, the primary objective is to assign a perfect cube number to each edge of a graph. This study focuses on analyzing a finite, undirected graph $G = (V, E)$ with p vertices and q edges. Notations used include $V(G)$ for the vertex set and $E(G)$ for the edge set of graph G. Standard graph theoretic notations are employed: P_n denotes a path with n vertices, C_n signifies a cycle with n vertices, $K(1, n)$ represents a star graph with $n+1$ vertices, and $T_{m,n}$ denotes a Tadpole graph. Terminology and notations follow established conventions, particularly those set by Harary.

Definition 1.1: A chord is an edge that connects two vertices of a cycle that are not contiguous.

Definition 1.2: We designate two chords in a cycle as twin chords if they combine to form a triangle with one of the cycle's edges. The graph $C_{n,p}$ comprises a cycle C_n with two twin chords and cycles C_p, C_3, and C_{n+1-p} without chords created by the edges of C_n. Furthermore, the graph Cn,p is for positive integers n and p, where three is smaller than or equal to n minus 2.

Definition 1.3: graph of tadpoles Combining a vertex of the cycle graph C_m with one of the leaves of the path graph P_n results in the formation of a graph denoted by the notation $T_{m,n}$.

Definition 1.4: Let $G = (p, q)$ be a graph. A bijection $f: V(G) \to \{0,1,2,\ldots\ldots\ldots, p-1\}$ is called sum perfect cube labeling of G, if the induced function $f*: E(G) \to N$ by $\boldsymbol{f*(uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2 f(v)}$ is injective, for all $u, v \in V(G)$. A graph that admits sum perfect cube (S.P.C.) labeling is called a sum perfect cube graph.

Literature Survey

In 2009, the investigation into Square Sum Graphs began, initiated by V. Ajitha *et al*. The authors demonstrated that certain graphs can be represented as square sums. Later, K. Germina and colleagues (2013) identified several other graphs as Square Sum graphs. The term "square difference graphs" was introduced by J. Shiama (2012), who also discovered graphs exhibiting square differences. S. Sonchhatra and G. Ghodasara (2016) defined a specific labeling known as Sum Perfect Square Labeling. They subsequently discovered various new Sum Perfect Square Graphs in 2016 and 2017, exploring different graph operations and graphs related to snakes. The concept of representing any number as a product of its prime factors' powers allows identifying numbers as perfect cubes if the sum of their prime factors is a multiple of three. In this study,

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a novel labeling scheme is developed to uniquely assign a perfect cube number to each edge of a graph. This labeling, termed sum-perfect cube labeling, characterizes graphs where every edge is labeled exclusively with perfect cube numbers, distinguishing them from others in the literature.

Main results

Lemma 3.1:P_nIs sum perfect cube graph, for all $n \geq 2, n \in N$.

Proof: Let $\{v_1, v_2, \ldots\ldots, v_n\}$be the route P$_n$ consecutive vertices. We define $f: V(P_n) \to \{0,1,2,\ldots\ldots, n-1\}$ as $f(v_i) = i - 1, 1 \leq i \leq n$.

Injectivity: When function f increases to i, we get $f(v_i) < f(v_{i+1})$ and so $f *: (E(G)) \to N$. So $f *: (E(G)) \to N$ is injective. Hence P_nIs the sum ideal cube graph for every $n \in N, n \geq 2$.

Lemma 3.2: Star $K_{1,n}$is the sum ideal cube graph for every $n \in N$.

Proof: Let $\{v_0, v_1, \ldots\ldots, v_n\}$represent the succession of vertices of the star $K_{1,n}$, where v_0 is the apex vertex. We define$f: V(K_{1,n}) \to \{0,1,2,\ldots\ldots, n\}$ as $f(v_0) = 0$ and $f(v_i) = i, 1 \leq i \leq n$."

Injectivity:Because the value of function f is growing in terms of $if or any i, m \in N, with i < m, we have f(v_i) < f(v_m)$, and so $f * (v_0 v_i) < f * (v_0 v_m), 1 \leq i, m \leq n$. So $f *: (E(G)) \to N$ is injective. Hence $K_{1,n}$ is sum perfect cube graph, for all $n \in N$.

Theorem 3.1: Every tree is a sum-perfect cube graph.

Proof: Let $\{v_0, v_1, \ldots\ldots, v_n\}$ as the T-letter sequence of vertices in a tree. Permit us to imagine Tas as a root-bearing tree. Assume that v0 represents the root vertex of T. V0 at vertex level 0. Let's$\{v_1, v_2, \ldots\ldots, v_k\}$be the level's subsequent vertices1, $1 \leq k \leq n (k \in N)$. $\{v_{k+1}, v_{k+2}, \ldots\ldots, v_t\}$be the successive vertices at level 2, $k + 1 \leq t \leq n (t \in N)$ and so on. We define $f: V(T) \to \{0,1,2,\ldots\ldots, n\}$ by $f(v_i) = i, 0 \leq i \leq n$.

Injectivity: At each consecutive level $l_m and l_{m+1}$, $m \in N \cup \{0\}, we get f(v_i) < f(v_j)$, for all $i and j$, where vertex v_i is at level l_m and vertex v_j is at level l_{m+1}. So $f *: (E(T)) \to N$ is injective. Hence, the tree is a perfect cube graph.

- A diagram displaying a tree with 15 vertices For a clearer understanding of the stated labeling scheme in Figure 1.

Theorem 3.2: $\boldsymbol{C_n}$Is the sum ideal cube graph for every $\boldsymbol{n \geq 3}$.

Proof: Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be the successive vertices of the cycle C_n. We define $f: V(C_n) \to \{0,1,2,\ldots\ldots, n-1\}$

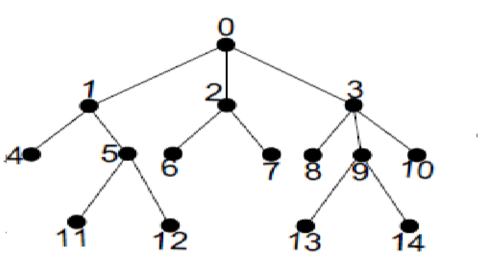

Figure 1: A tree with 15 vertices is a perfect cube graph

as follows:

$$f(v_i) = \begin{cases} 2i-2,; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n \end{cases}.$$

Let$f *: E(G) \to N$ by

$f * (uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2 f(v)$, $\forall u, v \in E(C_n)$.

Injectivity

For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ since $f$is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get

$f * (v_i v_{i+1}) < f * (v_{i+1} v_{i+2}), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$.

Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, as $f$is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$and so we get $f * (v_i v_{i+1}) > f * (v_{i+1} v_{i+2}), \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$.

Additionally, we see that $f * (v_1 v_n) = 1$, It appears on the graph's edges the tiniest.

Additionally $f * (v_{\left\lceil \frac{n}{2} \right\rceil} v_{\left\lceil \frac{n}{2} \right\rceil + 1})$ The graph's edge label is the highest. Right now $f(v_t)$is equal for all t, $1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil$ and $f(v_k)$is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n - 1$, we get $f * (v_t v_{t+1}) \neq f * (v_k v_{k+1})$. So $f *: (E(C_n)) \to N$is injective. Hence C_nIs the sum ideal cube graph, for every $n \geq 3, n \in N$.

Theorem 3.3: Cycle C_nA perfect cube graph with one chord is available for all $n \geq 4, n \in N$.

Proof: Let Gbe the chord-only cycle. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be the cycle's subsequent vertices C_n and $e = v_2 v_n$Be the chord of the cycle C_n. The vertices $v_1, v_2 and v_n$Form a triangle in C_nWith chord e. We define $f: V(C_n) \to \{0,1,2,\ldots\ldots, n-1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-2,; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n \end{cases}.$$

Let $f *: E(G) \to N$ by

$f * (uv) = [f(u)]^3 + [f(v)]^3 + 3f(u)f(v)^2 + 3f(u)^2 f(v)$, $\forall u, v \in E(G)$.

Injectivity

The chord $e = v_2 v_n$It is labeled by 27, distinct from all other edge labels as it can only be induced by the vertex labels 0,3 and 1,2. However, the pair 0,3 will never meet after labeling the vertex v_1 by one and v_n by 2. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, since f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$and so we get $f * (v_i v_{i+1}) < f * (v_{i+1} v_{i+2}), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, as $f$is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$and so we get $f * (v_i v_{i+1}) > f * (v_{i+1} v_{i+2}), \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$. Also

$f*(v_1v_n) = 1$, which, out of all the edge labels in the graph, is distinct and the smallest.

Additionally $f*(v_{\lceil \frac{n}{2} \rceil} v_{\lceil \frac{n}{2} \rceil + 1})$, which, being the highest edge label in the graph, sets it apart from all other edge labels. Right now $f(v_t)$ is even for each t, $1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n - 1$, we get $f*(v_tv_{t+1}) \neq f*(v_kv_{k+1})$. Therefore, each edge label is unique. Therefore C_n A perfect cube graph with one chord is available for all $n \geq 4$ with

Theorem 3.4: Cycle C_n Twin chords are the ideal cube graph in sum for everyone $n \geq 5$.

Proof: Let G be the dual-chord cycle. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be the cycle's subsequent vertices C_n and $e_1 = v_2v_n$ and $e_2 = v_3v_n$ Be the two chords of the cycle. C_n.

We define $f: V(C_n) \to \{0,1,2,\ldots\ldots, n-1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-2,;1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n \end{cases}.$$

Injectivity

The chord $e_1 = v_2v_n$ It is labeled by 27, which is distinct from all other edge labels as only the vertex labels 0,3, and 1,2 may cause this edge label to occur; yet, after labeling the vertex. v_1 by one and v_n By 2, the pair 0,3 will never cross paths with one another; the same justification holds for $e_2 = v_3v_n$, which is labeled by the label 125. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, since f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f*(v_iv_{i+1}) < f*(v_{i+1}v_{i+2})$, $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, as f Is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f*(v_iv_{i+1}) > f*(v_{i+1}v_{i+2})$, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$. Also $f*(v_1v_n) = 1$, which, out of all the edge labels in the graph, is distinct and the smallest.

Further $f*(v_{\lceil \frac{n}{2} \rceil} v_{\lceil \frac{n}{2} \rceil + 1})$, the tallest edge label on the graph, stands out from all other edge labels. Now $f(v_t)$ is even for each t, $1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n - 1$, we get $f*(v_tv_{t+1}) \neq f*(v_kv_{k+1})$. Consequently, each edge label is unique. Hence C_n A perfect cube graph with twin chords is available for all $n \geq 5$.

- The labeling pattern we specified in Theorems 3.2, 3.3, and 3.4 is justified by the information in Figure 2.

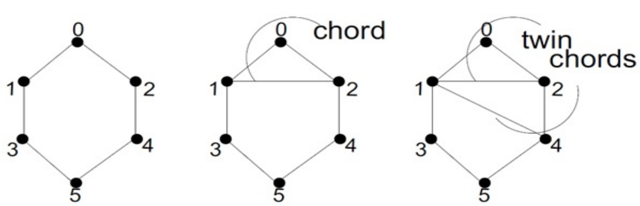

Figure 2: Cycle, with one chord, twin chords in a perfect cube graph

Theorem 3.5: K_n It is a perfect cube graph with sums, for $n < 4, n \in N$.

Proof: For $n = 1 and 2$, According to Theorem 1, graphs are a subset of trees, and they are particular instances of trees, K_1 and K_2 are graphs of the sum perfect cube cube. For $n = 3$, K_3 is the cycle C_3, and hence, as per Theorem 2, K_3 is a perfect cube graph. For $n = 4$, The vertices of the graph are labeled when we encounter the pair 1,2 and 0,3, for which we have identical induced edge labels. This occurs when we are at the K_4 position. As a result, the sum perfect cube graph K_4 is not it. For everyone $n \geq 4$, as $K_4 \subset K_5 \subset \ldots\ldots\ldots\ldots \subset K_n$, and so K_n It is not a perfect cube graph for all $n \geq 4$.

Theorem 3.6: Tadpole $T_{m,n}$ For any positive values, the sum perfect cube graph is idealintegers $mandn$.

Proof: Let $\{v_1, v_2, \ldots\ldots, v_m\}$ be the successive vertices of the cycle C_m and let $\{v_{m+1}, v_{m+2}, \ldots\ldots, v_{m+n}\}$ be the successive vertices of a path P_n in tadpole $T_{m,n}$. Let $e = v_mv_{m+1}$ be the bridge in tadpole $T_{m,n}$, where v_m It may be defined as the vertex with the highest label corresponding to the cycle. C_n and v_{m+1} is the vertex with the smallest label corresponding to the path P_n.

We define $f: V(T_{m,n}) \to \{0,1,2,\ldots\ldots, m+n-1\}$ as follows:

Case 1: m is even.

$$f(v_i) = \begin{cases} m - 2i, 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, \\ 2i - m - 1, \left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m' \\ i - 1, m + 1 \leq i \leq m + n \end{cases}$$

Injectivity

For $1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$, since f is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f*(v_iv_{i+1}) > f*(v_{i+1}v_{i+2})$, $1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1$. For $\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m + n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f*(v_iv_{i+1}) < f*(v_{i+1}v_{i+2})$, $\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m + n$. Moreover, for Also $f*(v_{\lceil \frac{m}{2} \rceil} v_{\lceil \frac{m}{2} \rceil + 1}) = 1$, What is the edge label that is the smallest of all the edge labels in this graph? In addition, $f*(v_1v_m)$ is the label with the highest edge in the cycle C_m As well as under the plan that we established, for $m + 1 \leq i \leq m + n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f*(v_iv_{i+1}) < f*(v_{i+1}v_{i+2})$. Now $f(v_t)$ is even for each t, $1 \leq t \leq \left\lceil \frac{m}{2} \right\rceil$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{m}{2} \right\rceil + 1 \leq k \leq m$, we get $f*(v_tv_{t+1}) \neq f*(v_kv_{k+1})$. So, all the edge labels are distinct. Hence $f*: (E(T_{m,n})) \to N$ is injective in this case."

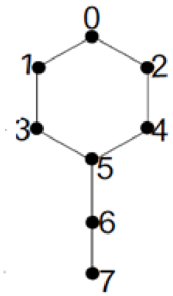

Figure 3: Tadpole in sum perfect cube Labeling

Case 2: m is odd.

$$f(v_i)=\begin{cases} \boldsymbol{m-2i-1, 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor,} \\ \boldsymbol{2i-m-1, \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m-1'} \\ \boldsymbol{i-1, m+1 \le i \le m+n} \end{cases}$$

Injectivity

For $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$, since $f$is decreasing in terms of i,
$f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get
$f*(v_i v_{i+1}) > f*(v_{i+1} v_{i+2})$,$1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor - 1$.
For $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m+n$, as$f$is increasing in terms of i,
$f(v_i) < f(v_{i+1}) < f(v_{i+2})$and so we get
$f*(v_i v_{i+1}) < f*(v_{i+1} v_{i+2}), \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m+n$.

Moreover, for Also $f*(v_{\left\lfloor \frac{m}{2} \right\rfloor} v_{\left\lfloor \frac{m}{2} \right\rfloor+1}) = 1$, What is the edge label that is the smallest of all the edge labels in this graph? In addition,$f*(v_1 v_m)$serves as the most prominent edge label in the cycleC_mAs well as following the plan that we established, for $m+1 \le i \le m+n$, as $f$is increasing in terms of i,$f(v_i) < f(v_{i+1}) < f(v_{i+2})$and so we get $f*(v_i v_{i+1}) < f*(v_{i+1} v_{i+2})$. Now $f(v_t)$is even for each t, $1 \le t \le \left\lfloor \frac{m}{2} \right\rfloor$ and $f(v_k)$is odd for each k, $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le k \le m$, we get $f*(v_t v_{t+1}) \ne f*(v_k v_{k+1})$.This ensures that each of the edge labels is unique. As a result $f*:(E(T_{m,n})) \to N$is injective in this case. Tadpole $T_{m,n}$ Is sum perfect cube graph for all positive integers $mandn$.

According to the labeling pattern specified in the accompanying Figure 3, Theorem no. 3.6 is applicable.

Conjecture

If every vertex in an odd graph (G) has a weird degree, every vertex v_i is constructed so that $d(v_i)$ is more significant than or equal to three. G is not a sum-perfect cube graph.

Conclusion

A graph's edges can only be identified by a single, distinct, perfect cube integer. The labeling presented in this research will open up a new concept for studying particular families of graphs that include many graphs. Several new theorems have been shown in this study, all connected to the recently proposed labeling. When everything is said and done, one hypothesis has been proposed, which may be considered an open issue.

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