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RESEARCH ARTICLE

A bivariate replacement policy (*T*, *N*) under partial product process

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Abstract

Considering an extreme shock maintenance model for a degenerative simple repairable system, explicit expressions for the long-run average cost under the bivariate replacement policy (*T*, *N*) has been obtained. The existence of the optimal value of (*T*, *N*) has been deduced. A numerical example is included to illustrate the theoretical results.

Keywords: Partial product process, Replacement policy, Shock models.

AMS subject classification: 60K10, 90B25

Introduction

In recent research efforts to optimize maintenance and replacement strategies for deteriorating systems under stochastic influences, the partial product process has emerged as a crucial mathematical tool. This process, first described in detail by Babu *et al.* (2023) investigates the probabilistic properties of a sequence of non-negative random variables characterized by a specific distribution function evolving through a multiplicative constant. This work lays the groundwork by not only detailing the process but also establishing several limit theorems that are essential for understanding the system's behavior over time.

Building on this fundamental understanding, Babu (2023) explored the practical implications of the suffering partial product process in a maintenance model for systems random shocks. The model uniquely incorporates the counting process of random shocks to determine the optimal replacement policy that minimizes long-term costs, presenting a significant advance in the maintenance of systems exposed to unpredictable environments.

Further elaboration on replacement strategies is presented by Raajpandiyan *et al.* (2022). This work extends the concept to a broader set of conditions, providing an optional replacement policy for systems that may undergo partial repairs, thus offering a nuanced approach to the maintenance of semi-repairable systems.

Raajpandiyan *et al.* (2023) also delve into the complexities of managing systems facing extreme shocks in their paper on a bivariate replacement policy. Here, they quantify the long-run costs associated with different strategies and compare the efficacies of bivariate and univariate policies, thus broadening the scope of replacement policy optimization.

Additionally, Faizanbasha and Rizwan (2024) introduced an innovative approach by incorporating the geometric point process to determine optimal replacement times for coherent systems. Their study not only fills a gap in the understanding of event dynamics and distributions but also highlights the practical implications of the process in optimizing maintenance schedules for critical components in high-stake industries like aviation and wind energy.

Moreover, Babu *et al.* (2019) explored a δ-shock maintenance model for systems with imperfect delayed repairs under partial product processes, assessing the long-run costs to find optimal replacement times. Govindaraju and Rajendiran (2020) investigate optimal maintenance policies for systems subjected to random shocks, focusing on repair times influenced by partial product processes. Hussainy and Shabeer (2021) discuss stochastic modeling in reliability systems, highlighting its role in enhancing

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maintenance decision-making for repairable systems with negligible downtime.

In this paper, we study the extreme shock maintenance model. We present the maintenance problem using the bivariate replacement policy. We also show that the bivariate optimal replacement policy $(T,N)^*$ is better than the univariate optimal replacement N^* and T^* policies.

Collectively, these studies deepend on our understanding of stochastic processes and maintenance modeling, significantly improving our ability to devise and implement cost-effective replacement policies in various industries. This expanding field promises great potential for further research and practical applications, potentially transforming maintenance strategies across diverse sectors.

Our study enhances existing research by integrating a bivariate replacement policy into an extreme shock maintenance model, which addresses limitations found in prior studies. Specifically, Rizwan and Bathmanaban (2019) focused primarily on theoretical definitions without exploring cost optimization in extreme conditions. Govindaraju and Kumar (2020) identified an optimal replacement strategy for degenerative systems but did not consider the complexities introduced by bivariate policies. Sun *et al.*'s (2022) model is limited by its reliance on delayed repair times and lacks comprehensive cost analysis for varying shock scenarios. Finally, Bohlooli-Zefreh *et al.* (2021) focus on reliability without fully optimizing the cost implications under different shock conditions. Our approach not only deduces the existence of an optimal policy but also provides explicit cost evaluations and a numerical example to illustrate practical applications.

Methodology

Model Description

Definition 2.1.

Partial product process. Let $\{X_n, n = 1,2,3,\ldots\}$ be a sequence of independent and non-negative random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_n, n = 1,2,3\ldots\}$ is called a partial product process if the distribution function of X_{k+1} is $F(\alpha_k x)(k = 1,2,3\cdots)$, where $\alpha_k > 0$ are real constants and $\alpha_k = \alpha_0\alpha_1\alpha_2\ldots\alpha_{k-1}$. In what follows, $F(x)$ denotes the distribution function of the non-negative random variable X_1.

Definition 2.2.

A partial product process is called a decreasing partial product process, if $\alpha_0 > 1$ and is called an increasing partial product process. If $0 < \alpha_0 < 1$. It is clear that if $\alpha_0 = 1$, then the partial product process is a renewal process.

Lemma 2.1.

Let $E(Y_1) = \mu, var(Y_1) = \sigma^2$. Then for $k = 1,2,3,\ldots. E(Y_{k+1}) = \frac{\mu}{\beta_0^{2^{k-1}}}$ and $var(Y_{k+1}) = \frac{\sigma^2}{\beta_0^{2^k}}$, where $\beta_0 > 0$.

Definition 2.3.

Bivariate replacement policy (T,N) It is a policy that replaces the system at T or at N-th failure since the last replacement, whichever occurs soon.

Model Assumptions

We make the following assumptions about the model for a simple degenerative reparable system subject to shocks.

- At time $t = 0$, a new system is installed. Whenever the system fails, it will be repaired. The system will be replaced by an identical new one, sometimes later.
- Once the system is operating, the shocks from the environment arrive according to a renewal process. Let $X_{ni}, i = 1,2,\ldots$ be the intervals between the $(i-1)$-st and i-th shock, after the $(n-1)$-st repair. Let $E(X_{11}) = \lambda$. Assume that X_{ni}, i=1,2,.... are independent and identically distributed random variables for all $n \in N$.
- Let $Y_{ni}, i = 1,2,\ldots$ be the sequence of the amount of damage produced by the i-th shock after the $(n-1)$ -st repair. Let $E(Y_{11}) = \mu$. Then $\{Y_{ni}, i = 1,2,\ldots\}$ are id sequences for all $n \in N$. If the system fails, it is closed so that the random shocks have no effect on the system during the repair time. In the n-th operating stage, that is, after the $(n-1)$-st repair, the system will fail, if the amount of the shock damage first exceed $\alpha_0^{2^{n-1}}M$, where $0 < \alpha_0 \le 1$ and M is a positive constant.
- Let $Z_n, n = 1,2,\ldots$ be the repair time after the n-th repair and $\{Z_n, n = 1,2,\ldots\}$ constitute a non-decreasing partial product process with $E(Z_1) = \delta$ and ratio β_0, such that $0 < \beta_0 < 1$. $N_n(t)$ is the counting process denoting the number of shocks after the $(n-1)$-st repair. It is clear that
$$E(Z_n) = \frac{\mu}{\beta_0^{2^{n-1}}}.$$
- Let r be the reward rate per unit time of the system when it is operating and c be the repair cost rate per unit time of the system and the replacement cost is R. The replacement time is a random variable Z with $E(Z) = \tau$.
- The sequences $\{X_{ni}, i = 1,2,\ldots\}, \{Y_{ni}, i = 1,2,\ldots\}, \{Z_n, n = 1,2,\ldots\}$ and Z are independent.
- The replacement policy (T,N) is adapted.

Results and Discussion

The Bivariate Replacement Policy (T, N)

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (*T*, *N*) policy. Let

$$L_n = \min\{l; Y_{nl} > \alpha_0^{2^{n-1}}M\}$$

and

$$W_n = \sum_{i=1}^{L_n} X_{ni}.$$

Thus L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P[L_n = k] = p_n q_n^{k-1}, k = 1,2,\ldots,$ where $p_n = P[Y_{nl} > \alpha_0^{2^{n-1}} M]$ and $q_n = 1 - p_n$.

We have $E(L_n) = \frac{1}{p_n}$. Since $\{X_{ni}, i = 1,2,\ldots\}$ and $\{Y_{ni}, i = 1,2,\ldots\}$ are independent, it is clear that L_n and $\{X_{ni}\}$ are independent. Now

$$E(W_n) = E\left(\sum_{i=1}^{L_n} X_{ni}\right)$$
$$= E(L_n)E(X_{n1})$$
$$= \frac{\lambda}{p_n}.$$

The distribution function of W_n is $F_n(\cdot)$

The working age T of the system at time t is the cumulative lifetime given by

$$T(t) = \begin{cases} t - V_n, & U_n + V_n \le t < U_{n+1} + V_n \\ U_{n+1}, & U_{n+1} + V_n \le t < U_{n+1} + V_{n+1}, \end{cases}$$

where $U_n = \sum_{i=1}^{n} W_i$ and $V_n = \sum_{i=1}^{n} Z_i$ and $U_0 = V_0 = 0$.

Let T_1 be the replacement time; in general for $n = 2,3,\ldots,$ let T_n be the time between the $(n-1)$-st replacement and the n-th replacement. Thus, the sequence $\{T_n, n = 1,2,\ldots\}$ forms a renewal process. A cycle is completed if a replacement is done. A cycle is actually the time interval between the installation of the system and the first replacement or the time interval between two consecutive replacements. Finally, the successive cycles, together with the cost incurred in each cycle, will constitute a renewal reward process.

The length of the cycle under the replacement policy (T,N) is

$$W = [T + \textstyle\sum_{n=1}^{\eta} Z_n]\chi_{(U_n>T)} + [\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n]\chi_{(U_n \le T)} + Z,$$

where $\eta = 0,1,2,\ldots N-1$ is the number of failures before the working age of the system exceeds T and $\chi_{(A)}$ denotes the indicator function.

The expected length of a cycle is

$$E(W) = E\left[T + \left[\sum_{n=1}^{\eta} Z_n\right]\chi_{(U_n>T)}\right] + E\left[\left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right]\chi_{(U_n \le T)}\right] + E(Z)$$

$$= E[T\chi_{(U_n>T)} + E[\textstyle\sum_{n=1}^{\eta} Z_n]\chi_{(U_n>T)}]$$
$$+E[E[\textstyle\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n]\chi_{(U_n \le T)}|U_n = u] + E(Z)$$

$$= T\overline{F}_N(T) + \lambda\left[F_1(T) + \textstyle\sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \times E[\chi_{(U_n \le T < U_n)}] + \int_0^T E[\sum_{n=1}^{N} W_n] u dF_N(u)$$
$$+ \textstyle\int_0^T [\sum_{n=1}^{N-1} E(Z_n)] dF_N(u) + \tau$$

$$= T\overline{F}_N(T) + \lambda\left[F_1(T) + \textstyle\sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \times P(U_N \le T < U_n) + \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_{N(u)}$$
$$+\mu\left[1 + \textstyle\sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] F_N(T) + \tau$$

$$= T\overline{F}_N(T) + \lambda\left[F_1(T) + \textstyle\sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \times [F_n(T) - F_N(T)] + \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u)$$
$$+\mu\left[1 + \textstyle\sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] F_N(T) + \tau$$

$$= T\overline{F}_N(T) + \lambda\left[F_1(T) + \textstyle\sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \times F_n(T) + \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u)$$
$$+\mu\left[1 + \textstyle\sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] + \tau. \quad (1)$$

Let $C(T,N)$ be the long-run average cost per unit of time under the bivariate replacement policy (T,N). By the renewal reward theorem, the long-run average cost per unit time under the replacement policy - (T,N) is given by

$$C(T,N) = \frac{\text{expected cost incurred in a cycle}}{\text{expected length of a cycle}}$$
$$= \frac{\begin{bmatrix} E\left[[c\sum_{n=1}^{\eta} Z_n - rT]\chi_{(U_n>T)}\right] + c_p E(Z) \\ +E\left[[c\sum_{n=1}^{N-1} Z_n - r\sum_{n=1}^{N} W_n]\chi_{(U_n \le T)}\right] + R \end{bmatrix}}{E(W)}. \quad (2)$$

Consider

$$E[\textstyle\sum_{n=1}^{\eta} X_k] = E[E(\sum_{k=1}^{\eta} X_k|\eta = n)]$$
$$= \textstyle\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\eta} E(X_k)\right) P(\eta = n)$$
$$= \textstyle\sum_{n=1}^{\infty} [E(x_1) + \sum_{k=2}^{\eta} E(X_k)] P(\eta = n)$$
$$= E(X_1)\textstyle\sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=2}^{n} E(X_k)\right) P(\eta = n)$$
$$= \lambda \textstyle\sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} E(X_{k+1})\right) P(\eta = n)$$
$$= \lambda \textstyle\sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{\lambda}{\alpha_0^{2^{k-1}}}\right) P(\eta = n)$$
$$= \lambda P(n = 1) + \lambda \textstyle\sum_{n=2}^{\infty} \left[1 + \sum_{k=1}^{n-1} \frac{1}{\alpha_0^{2^{k-1}}}\right] P(\eta = n)$$
$$= \lambda(F_1(T) - F_2(T)) + \lambda \textstyle\sum_{n=2}^{\infty} \left[1 + \sum_{k=1}^{n-1} \frac{1}{\alpha_0^{2^{k-1}}}\right](F_n(T) - F_{n+1}(T))$$
$$= \lambda F_1(T) + \lambda \textstyle\sum_{n=1}^{\infty} \frac{1}{\alpha_0^{2^{n-1}}} F_{n+1}(T)$$
$$= \lambda\left[F_1(T) + \textstyle\sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right]. \quad (3)$$

Now,

$$E[\textstyle\sum_{n=1}^{N-1} Z_n] = E[(\sum_{k=1}^{N-1} Y_k|\eta = n)]$$
$$= \left(\textstyle\sum_{k=1}^{N-1} E(Y_k)\right)$$
$$= [E(Y_1) + \textstyle\sum_{k=2}^{N-1} E(Y_n)]$$
$$= \left[\mu + \textstyle\sum_{k=2}^{N-1} \left(\frac{\mu}{\beta_0^{2^{k-2}}}\right)\right]$$
$$= \mu\left[1 + \textstyle\sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] \quad (4)$$

On substituting (1), (3) and (4) in equation (2), we obtain the following

Theorem 3.1.`

For the model described in section 2, under assumptions 2.1 to 2.7, the long-run average cost per unit time under the bivariate replacement policy (T,N) for a simple degenerative repairable system is given by

$$C(T,N) = \frac{\begin{bmatrix} c\lambda\left[F_1(T) + \sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] + c_p\tau + R - rT\overline{F}_N(T) \\ - r\sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \mu\left[1 + \sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] \end{bmatrix}}{\begin{bmatrix} T\overline{F}_N(T) + \lambda\left[F_1(T) + \sum_{n=1}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \times F_n(T) \\ + \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u) + \mu\left[1 + \sum_{n=2}^{N-1} \frac{1}{\beta_0^{2^{n-2}}}\right] + \tau \end{bmatrix}} \quad (5)$$

Deductions

The long run average cost $C(T,N)$ is a bivariate function in T and N. Obviously, when N is fixed, $C(T,N)$ is a function of T. For fixed $N=m$, it can be written as

$$C(T,N)=C_m(T), m=1,2,....$$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $C_m(T_m^*)$ is minimized. That is, when $N=1,2,...,m,...$, we can find T_1^*, T_2^*, T_3^*, T_m^*, respectively, such that $C_1(T_1^*), C_2(T_2^*),...,C_m(T_m^*),...$ are minimized. Because the total life -time of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So, we can determine the minimum of long-run average cost per unit of time based on $C_1(T_1^*), C_2(T_2^*),...,C_m(T_m^*),....$ Then, if the minimum is denoted by $C_n(T_n^*)$, we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$C((T,N)^*)=\min_N C_n(T_n^*)$$
$$=\min_N[\min_T C(T,N)]$$
$$\leq C(\infty,N)$$
$$\equiv C(N^*)$$

the optimal policy $(T,N)^*$ is better than the optimal policy N^*. Moreover, under some mild conditions, an optimal replacement policy N^* is better than the optimal policy T^*. so under the same conditions, an optimal policy $(T,N)^*$ is better than the optimal replacement policies N^* and T^*.

Numerical Example

In this section, we give an example to illustrate theoretical results. Assume that $\{X_i, i=1,2,3,...\}$ is a sequence of independent random variables and each X_i has an exponential distribution $\exp(\lambda_i)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, the probability density function of $\sum_{i=1}^{n} X_i$ is given by

$$f_n(t)=\begin{cases}(-1)^{n-1}\lambda_1\lambda_2\cdots\lambda_n \times \sum_{i=1}^{n}\frac{\exp(-\lambda_i t)}{\prod_{j=1,j\neq i}^{n}\lambda_{i-1}-\lambda_{j-1}} & ; x\geq 0\\ 0 & ; otherwise.\end{cases}$$

Let $\lambda_i=\frac{\lambda}{\alpha_0 2^{i-1}}$ for i = 1,2,3,.... Then the distribution $\sum_{i=1}^{n} X_i$ is

$$F_n(T)=(-1)^{n-1}\left(\frac{\lambda}{\alpha_0}\right)^n\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}}\times\sum_{i=1}^{n}\frac{1-\exp\left(-\frac{\lambda}{\alpha_0 2^{i-1}}T\right)}{\prod_{j=1,j\neq i}^{n}\left(\frac{\lambda}{\alpha_0 2^{i-1}}-\frac{\lambda}{\alpha_0 2^{j-1}}\right)}.$$

The distribution function of $\sum_{i=1}^{n} Y_i$ is

$$G_n(T)=(-1)^{n-1}\left(\frac{\mu}{\beta_0}\right)^n\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}}\times\sum_{i=1}^{n}\frac{1-\exp\left(-\frac{\mu}{\beta_0 2^{i-1}}T\right)}{\prod_{j=1,j\neq i}^{n}\left(\frac{\mu}{\beta_0 2^{i-1}}-\frac{\mu}{\beta_0 2^{j-1}}\right)}.$$

Let the parameter values be

$\lambda=75$ $c=9$ $R=50000$

$\mu=25$ $r=7$ $\tau=12$

$\alpha_0=1.05$ $\beta_0=0.95$ $c_p=4$

Table 1: Values of $C(T,N)$ against (T,N)

(T,N)	$C(T,N)$	(T,N)	$C(T,N)$
(310,15)	16.0256	(310,18)	−19.3215
(320,15)	16.0378	(320,18)	−31.5237
(330,15)	16.5462	(330,18)	−52.9824
(340,15)	16.7461	(340,18)	−79.8651
(350,15)	16.7573	(350,18)	−103.5436
(360,15)	16.8925	(360,18)	– 389.1743
(370,15)	16.9106	(370,18)	−189.2415
(380,15)	17.0254	(380,18)	−79.1432
(390,15)	17.5324	(390,18)	−23.4165
(400,15)	17.6655	(400,18)	17.1946
(310,16)	23.5487	(310,19)	18.4236
(320,16)	23.6402	(320,19)	18.5199
(330,16)	23.8945	(330,19)	18.5237
(340,16)	23.9988	(340,19)	18.6175
(350,16)	24.7235	(350,19)	18.7189
(360,16)	24.8259	(360,19)	18.8243
(370,16)	25.7514	(370,19)	18.8435
(380,16)	24.8013	(380,19)	18.8767
(390,16)	24.9988	(390,19)	18.8923
(400,16)	25.6256	(400,19)	19.1406
(310,17)	26.1243	(310,20)	21.1293
(320,17)	26.2233	(320,20)	−21.1764
(330,17)	26.3014	(330,20)	21.2147
(340,17)	26.4215	(340,20)	21.7985
(350,17)	26.5109	(350,20)	22.4356
(360,17)	26.5324	(360,20)	23.5243
(370,17)	26.6235	(370,20)	23.6075
(380,17)	26.7525	(380,20)	23.9106
(390,17)	26.8529	(390,20)	23.9258
(400,17)	26.9017	(400,20)	23.9987

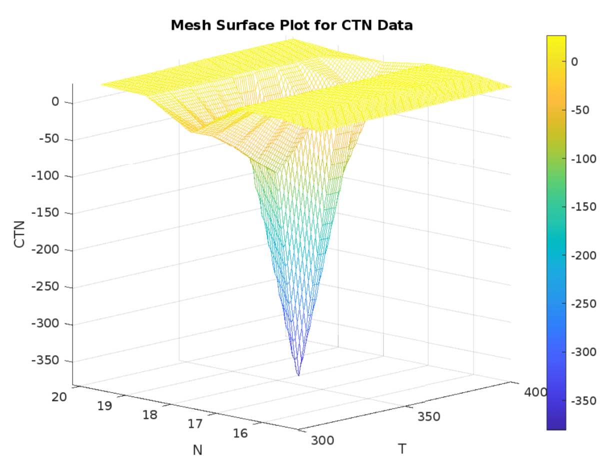

Figure 1: Plot of $C(T,N)$ against (T,N)

On substituting these values in equation (5) and overpassing numerical calculations, we arrive at $(T,N)^*=(360,18)$, so that $C(T,N)$ is minimum at $(T,N)^*$ and the long run average cost $C(T,N) = C(360,18) = -389.1743$ monetary units. The value of $C(T,N)$ for T ranging from 310 to 400 with step size 10 and N ranging from 15 to 20 with step size 1 are calculated and the results are plotted in Figure 1. The calculated values of $C(T,N)$ are given in Table 1.

Conclusion

In this paper, we have considered an extreme shock maintenance model for a degenerative simple repairable system. An explicit expression for the long-run average cost under the bivariate replacement policy (T,N) is derived. The existence of the optimal value of (T,N) is deduced. A numerical example is included to illustrate the theoretical results.

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